GRAPH

Graph $\equiv$ set + binary relation

$V$ - vertices

$E$ - edges

**Definition**

A **simple graph** $G = (V, E)$

$V \neq \emptyset$ set of vertices

$E$ set of unordered pairs of distinct vertices

**Example**

Computer net connected by telephone lines. Data flows both ways. No computer is connected to itself by telephone.
**MULTIGRAPH = GRAPH WITH MULTIPLE EDGES**

**DEF:** A multiGraph \( G = (V, E, \mathcal{f}) \)

- \( V \neq \emptyset \) SET OF VERTICES
- \( E \) SET OF EDGES
- \( \mathcal{f} \) FUNCTION FROM \( E \) TO THE SET OF UNORDERED PAIRS OF VERTICES

**EXAMPLE:** COMPUTER NET WITH MULTIPLE TELEPHONE LINES

A diagram showing vertices labeled \( a, b, c, 1, 2, 3, 4 \) connected by lines indicating edges.

\[ \mathcal{f}(1) = \mathcal{f}(2) = \{a, b\} \]
\[ \mathcal{f}(3) = \mathcal{f}(4) = \{b, c\} \]

Parallel edge's \( \mathcal{f}(e_1) = \mathcal{f}(e_2) \)

Still no loops, direction does not matter.

**AUTO, AIRPLANE CONNECTION MAPS**
**Pseudograph = Multigraph with loops**

**def a Pseudograph** $G = (V, E, \phi)$

- $V \neq \emptyset$ set of vertices
- $E$ set of edges
- $\phi$ vertices reading function from $E$ to set of unordered pairs of vertices not necessarily distinct.

**Example.** A computer net with multiple telephone lines, including self-connected for diagnostic purposes.
**Directed Graph** = Graph with directed edges

Let a **Directed Graph** $G = (V, E)$

- $V \neq \emptyset$: set of vertices
- $E \subseteq V \times V$: edges = ordered pairs of vertices

**Example:** Computer net with asymmetric lines, no multiple lines in the same direction.

**Directed Multigraph** $G = (V, E, \mathcal{f})$

- $V \neq \emptyset$: set of vertices
- $E$: set of edges
- $\mathcal{f}: E \rightarrow V^2$: reads initial and terminal vertices of a given edge
def ADJACENT VERTICES u, v ∈ V such that (u, v) ∈ E, E = E(u, v) connects u, v. E is incident with u, v. u, v are endpoints of E.

def DEGREE OF A VERTEX = # OF EDGES INCIDENT WITH IT (A LOOP CONTRIBUTES TWICE!)

AN ISOLATED VERTEX

\[\text{deg}(a) = 2\]
\[\text{deg}(b) = 6\]
\[\text{deg}(c) = 3\]
\[\text{deg}(d) = 3\]
\[\text{deg}(e) = 0\]

TH. (THE HANDSHAKING THEOREM) \[G = (V, E)\] PSEUDOGRAPH (UNDIRECTED)
\[2e = \sum_{v \in V} \text{deg}(v)\]
\[e = |E|\]

PROOF. EVERY EDGE IS COUNTED TWICE IN E.
TH. AN UNDIRECTED (PSEUDO)GRAPH HAS AN EVEN NUMBER OF VERTICES OF ODD DEGREE.

PROOF. LET $V = V_1 + V_2$ WHERE $V_1 =$ VERTICES WITH EVEN DEGREES, $V_2 =$ ODD.

$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$

EVEN         EVEN

$\Rightarrow \sum_{v \in V_2} \deg(v)$ is EVEN $\Rightarrow |V_2|$ is EVEN.