LECT 21 10/28/99

PROBABILITY III.

RANDOM VARIABLE IS A FUNCTION FROM S TO \( \mathbb{R} \) [REAL NUMBERS]

NOTATION: \( X(s) \) \( s \in \Omega \)

EXAMPLE: A coin is flipped 5 times
\( X(s) = \) the number of heads in \( s \)

\( n = 3 \)
\( X(\text{TTT}) = 0 \)
\( X(\text{THH}) = X(\text{HHT}) = X(\text{HTH}) = 1 \)
\( X(\text{HHT}) = X(\text{HHH}) = 2 \)
\( X(\text{HHH}) = 3 \)

EXAMPLE: \( X(s) = \) the sum of numbers that appear when a pair of dice is rolled

\( X(1,1) = 2 \)
\( X(2,1) = X(1,2) = 3 \)
\( X(5,6) = X(6,5) = 11 \)
\( X(6,6) = 12 \)

NOTE: RANDOM VARIABLE IS NEITHER VARIABLE NOR RANDOM

\(-127-\)

\[ E(X) = \sum_{i=1}^{n} C(n, k) \cdot p^k \cdot q^{n-k} = np \cdot \sum_{i=0}^{n} C(n, i) \cdot p^i \cdot q^{n-i} \]

BY THE BINOMIAL FORMULA = \( (pq)^n \) = \( 1 \)

\[ X, Y \] ARE RANDOM VARIABLES ON A SPACE \( \Omega \)
\[ E(X+Y) = E(X) + E(Y) \]

PROOF: \( E(X+Y) = \sum p(x) (X(s)+Y(s)) \sum E(p(x)+Y(s)) + E(p(x)) \sum Y(s) \sum E(Y) \)
\[ E(cX) = \sum p(x) c \cdot x \sum E(p(x) \cdot x) = c \cdot E(x) \]

\[ E(aX+b) = E(aX) + E(b) = a \cdot E(x) + E(b) = a \cdot E(x) + b \]

\[ E(aX+b) = E(aX+b) = a \cdot E(x) + b \]

\[ E(b) = \sum p(x) \cdot b \sum E(p(x)) = b \cdot \sum E(p(x)) = b \cdot 1 = b \]

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The variance of $X$ is a measure of how widely $X$ is distributed around its expected value, $E(X)$. The above formula:

$$V(X) = E((X - E(X))^2)$$

is expressed in a similar way.

**Proof:**

$$V(X) = E((X - E(X))^2) = E(X^2) - 2E(X)E(X) + E(X)^2$$

$$= E(X^2) - 2EX^2 + E^2 X$$

$$= E(X^2) - E^2 X$$

**Example:** Bernoulli trial $X(\omega) \in \{0, 1\}$, $P(T = 1) = p$. Note that $X^2(\omega) = X(\omega) = p = E(X)$. The variance of $X$ is $E(X^2) - [E(X)]^2$.

$$V(X) = p - p^2 = p(p-1)$$

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**Average-case computational complexity**

1. Assign a probability $P(\omega)$ to each possible input value $\omega$.

2. $E(\omega)$ is the average-case complexity.

**Example:** The average-case complexity of the linear search algorithm.

- Numbers: a sample number $X$.
- $2i+1$ comparisons if $X$ is inside the list.
- $P = \text{probability that } X \text{ is in the list}$
- $Q = 1-P$ not $X$
- $P_m = \text{probability that } X \text{ is not in the list}$
- $E = P(\omega) + P(\omega) + \ldots + (2 \times P(\omega) \times P(\omega) + \ldots + (2 \times P(\omega) + (2 \times P(\omega) = \sum_{i=0}^{n-1} (2 \times P(\omega)) = 2 \times P(\omega)$

If $P(\omega) = 0.5$, $E = n+2$ (worst case).