CLASSICAL PROBABILITY

Sample Space = Set of possible outcomes of an experiment

Event = Subset of the sample space

Example: An experiment: Two dice are rolled

The sample space = \( \{1,2,3,4,5,6\}^2 = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), (3,1), (3,2), \ldots, (3,6), \ldots, (6,1), \ldots, (6,6)\} \)

36 pairs total

An event: The sum on the two dice is five

\( \{(1,4), (2,3), (3,2), (4,1)\} \) 4 pairs

The main assumptions under which the classical definition of probability operates

1. Sample space \( S \) is finite

2. All outcomes in \( S \) are equally likely

Def (Laplace): The probability of an event \( E \)

\[ p(E) = \frac{|E|}{|S|} \]

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Example: The probability that when two dice are rolled, the sum is 5.

\[ P = \frac{4}{36} = \frac{1}{9} \]

Example: An urn contains three blue balls and five red balls. What is the probability that a ball chosen is blue?

\[ |S| = 8; \quad |E| = 3, \quad P_B = \frac{|E|}{|S|} = \frac{3}{8} \]

Example (lottery): What is the probability to pick the correct six numbers out of 40?

Sample space = all possible 6-combinations out of 40.

\[ |S| = \frac{40!}{6!34!} = 3838380 \]

Event = one winning combination \[ |E| = 1 \]

\[ P = \frac{|E|}{|S|} = \frac{1}{3838380} \]

Example (poker): Find the probability that a hand of five cards contains four cards of one kind.

\[ S = \text{set of all hands}; \quad |S| = \binom{52}{5} \]

\[ |E| = (\text{by the product rule}) \quad \# \text{ of ways to pick a kind} \times \# \text{ of ways to pick the fifth card} \]

\[ = \binom{13}{1} \cdot \binom{48}{4} \]

\[ P = \frac{\binom{13}{1} \cdot \binom{48}{4}}{\binom{52}{5}} = \frac{13 \cdot 48}{2598960} \approx 0.00024 \]
Complementary Event: \( E = S - e \)

\[ E = |S| - |e| \]

Hence, \( P(E) = 1 - P(e) \)

**Proof:** \( P(e) = \frac{|e|}{|S|} = \frac{|S| - |e|}{|S|} = \frac{|S|}{|S|} - \frac{|e|}{|S|} = 1 - P(E) \)

**Example:** Find the probability that an integer \( x \) (0 ≤ x ≤ 999999) has at least one 9 in its decimal expansion.

\( E = \"x has no 9\"; \text{ by the product rule:} \)

Six decimal positions, digits \( \{0, 1, \ldots, 8\} \) for each position

\[ |E| = 0.9 \cdot 0.9 \cdot 0.9 \cdot 0.9 = 0.9^6 \]

\[ |S| = \text{total # of } x \text{'s from } 0 \text{ to } 999999 = 10^6 \]

\[ P(e) = \frac{0.9^6}{10^6} = (0.9)^6 = 0.53; \text{ } P(E) = 1 - 0.53 = 0.47 \]

Example (Tossing a Coin): A sequence of 10 bits is randomly generated. What is the probability that at least one of them is 1?

\( E = \"all bits are 0's\"; \)

\[ |E| = 1; \text{ } |S| = 2^{10} \]

\[ P(e) = 1 - P(E) = 1 - \frac{1}{2^{10}} = \frac{1023}{1024} \]
Th. \( P(E \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \)

**Proof.**
\[
|E \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|
\]
\[
P(E \cup E_2) = \frac{|E \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{s} = P(E_1) + P(E_2) - P(E_1 \cap E_2)
\]

**Example.** Positive integers \( \leq 100 \) are randomly selected. What is the probability that a number is divisible by 2 or 3?

- **\( E_1 \)**: X is divisible by 2 \( \quad P_1 = \frac{50}{100} \)
- **\( E_2 \)**: X is divisible by 3 \( \quad P_2 = \frac{33}{100} \)
- **\( E_1 \cap E_2 \)**: X is divisible by both 2 and 3 \( \quad P_3 = \frac{16}{100} \)

\[
P = P_1 + P_2 - P_3 = \frac{50}{100} + \frac{33}{100} - \frac{16}{100} = \frac{50 + 33 - 16}{100} = \frac{67}{100}
\]

HW. 4.4: 8 14 24 31 32