1. (5 points each) The English alphabet contains 5 vowels and 21 consonants.
   a) How many strings of 6 lowercase English letters contain exactly 2 vowels if each letter
   may be used as often as you like?

   There are $C(6, 2)$ ways to choose the 2 slots for the vowels, then there are 5 choices for
   each vowel and 21 choices for each consonant, so there are $C(6, 2)5^221^4$ total possible strings.

   b) How many strings of 6 lowercase English letters contain exactly 2 vowels if no letter
   is allowed to be used more than once?

   Again there are $C(6, 2)$ choices for the slots for the vowels, then $P(5, 2)$ choices for the
   vowels and $P(21, 4)$ choices for the consonants, so a total of $C(6, 2)(5 \cdot 4)(21 \cdot 20 \cdot 19 \cdot 18)$
   possible strings.

2. (5 points each) a) What’s the minimum number of people that must be chosen to be
   sure that at least 2 have the same first initial?

   Since there are 26 letters, the Pigeonhole principle implies that if we have 27 people then
   at least 2 must have the same first initial.

   b) What’s the minimum number of people that must be chosen to be sure that at least 3
   have the same birth month and and were born on the same day of the week (Sat, Sun, Mon,
   etc)?

   Now there are $(12)(7) = 84$ slots, so by the Pigeonhole principle, we want to choose $n$
   so that $[n/84] = 3$. The smallest $n$ for which this is true is $n = 2(84) + 1 = 169$.

   c) Suppose there are 50 people with ages between 1 and 98 (1 and 98 are allowed). Show
   that either there are 2 people with the same age or two whose ages are consecutive integers.

   Create 49 slots by grouping the integers into consecutive pairs starting with 1: {1, 2}, {3, 4}, \ldots {97, 98}. Since there are 50 people with ages contained between 1 and 98, the Pigeon-
   hole principle implies that at least one slot contains at least 2 people. For this slot, either
   the two people have the same age, or their ages are consecutive integers.

3. (5 points each) a) How many bit strings contain exactly 6 0’s and 9 1’s if every 0 must
   be immediately followed by a 1?

   Think of the string ‘01’ as a unit, in which case there are 6 units of ’01’ and 3 more units
   of ‘1’. Hence there are 9 total slots to be filled, and 3 of these are filled with ‘1’, so there are
   $C(9, 3)$ total possible strings.
b) How many solutions are there to the equation \( x_1 + x_2 + x_3 + x_4 = 35 \), if each \( x_j \) is a positive integer (i.e., 1 or bigger)?

This is a stars and bars problem with 4 baskets and 35 stars. However, we need to make sure that each basket has at least one star, so first we set aside 4 stars - one for each basket. Then we have 31 stars and 4 baskets, so there are \( C(31 + 4 - 1, 3) = C(34, 3) \) possible ways to assign the stars, so this is the number of solutions.

4. (5 points each) a) Find the number of elements in \( A_1 \cup A_2 \cup A_3 \) if there are 100 elements in each set and there are 25 common elements in each pair and 10 elements in the intersection of all 3 sets.

By the inclusion-exclusion principle, we need have \( |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3| \). Using the information given, we get
\[
100 + 100 + 100 - (25 + 25 + 25) - 10 = 235.
\]

b) Suppose an experiment consists of choosing one of the elements of \( A_1 \cup A_2 \cup A_3 \) at random with equal probability (where these three sets have the number of elements and intersections as given in part a). Are the events \( A_1 \) and \( A_2 \) independent?

In order for the events to be independent, we would need \( P(A_1 \cap A_2) = P(A_1)P(A_2) \). From part (a), we know there are 235 elements in total, and that \( A_1 \) and \( A_2 \) each have 100 elements, while \( A_1 \cap A_2 \) has 25 elements. So the question reduces to comparing \( 25/235 = .10638 \) and \( (100/235)^2 = .181077 \). Thus the two values are not the same and so the two events are not independent.

5. (5 points each) The dice in this question are standard six-sided fair dice, and rolling a number with more than one die refers to the sum of the numbers showing on the dice.

a) What is the probability of rolling a 5 with 2 dice?

There are 4 ways to roll a 5: (1,4), (2, 3), (3, 2) and (4, 1). Since there are 36 total equally likely possibilities, the probability of rolling a 5 is \( 4/36 = 1/9 \).

b) What is the probability of rolling a 5 with 3 dice?

There are 6 ways to roll a 5: (1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1) and (3, 1, 1). Since there are \( 6^3 \) total equally likely possibilities, the probability of rolling a 5 is \( 6/6^3 = 1/36 \).

6. (7 points) What is the expected sum that appears on 2 dice, where each of the dice is biased so that a 3 appears with probability .3 and the other 5 numbers all have equal
probability? (For this problem, do the arithmetic to determine the final probability as a number).

Let \( X_j \) be the random variable that records the value of die \( j \) for \( j = 1, 2 \). Then since each of the other numbers appears with probability \((1 - .3)/5 = .14\), we have

\[
E(X_j) = .14(1) + .14(2) + .3(3) + .14(4) + .14(5) + .14(6) = .14(18) + .9 = 3.42
\]

Since \( X_1 \) and \( X_2 \) are independent (since the 2 die are independent), we have \( E(X_1 + X_2) = E(X_1) + E(X_2) = 6.84 \).

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7. a) (8 points) Find a recurrence relation for the number of binary strings of length \( n \) that contain 2 consecutive 0's. Also, give the initial conditions for this recurrence relation.

Let \( a_n \) denote the number of strings of length \( n \) with 2 consecutive 0's. Given a string of length \( n \) with 2 consecutive 0's, either it ends in 1 and the first \( n - 1 \) bits have 2 consecutive 0's (and there are \( a_{n-1} \) strings of this type), or it ends in 0. If it ends in 0, then the \( n - 1 \)st bit may be a 1, in which case the first \( n - 2 \) bits must have 2 consecutive 0's (and there are \( a_{n-2} \) strings of this type), or the \( n - 1 \)st bit is a 0, in which case the first \( n - 2 \) bits could be anything (and there are \( 2^{n-2} \) strings of this type). Hence \( a_n = a_{n-1} + a_{n-2} + 2^{n-2} \) for \( n \geq 3 \). For a string of length 1, there are no strings with 2 consecutive 0's, and for a string of length 2, there is exactly 1, so \( a_1 = 0 \) and \( a_2 = 1 \).

b) (5 points) How many strings of length \( n = 6 \) have 2 consecutive 0's?

From part (a), \( a_1 = 0 \) and \( a_2 = 1 \). Then using the recurrence relation, we have

\[
\begin{align*}
a_3 &= 1 + 0 + 2 = 3 \\
a_4 &= 3 + 1 + 2^2 = 8 \\
a_5 &= 8 + 3 + 2^3 = 19 \\
a_6 &= 19 + 8 + 2^4 = 43
\end{align*}
\]

Hence there are 43 such strings.