**Exercise 1** (10 points) A sequence $X_n$ is defined recursively by

$X_0 = a$

$X_1 = b$, where $a, b$ are reals

$X_n = X_{n-1} + X_{n-2}$, for $n > 1$.

Prove by induction, that $X_n = b \cdot f_n + a \cdot f_{n-1}$ for all $n > 0$, where $f_n$ is the $n$th Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_n = f_{n-2} + f_{n-1}$ for $n > 1$).

**Solution:** We shall prove by induction that $X_n = b \cdot f_n + a \cdot f_{n-1}$ for all $n > 0$ (*).

First of all, this is true for $n = 1$ since $X_1 = b = b \cdot 1 + a \cdot 0 = b \cdot f_1 + a \cdot f_0$.

This is true for $n = 2$ also since $X_2 = X_1 + X_0 = b \cdot 1 + a \cdot 1 = b \cdot f_2 + a \cdot f_1$.

Given an arbitrary $n > 1$, let us assume (*) is true for $n - 1$ and for $n$.

$$X_{n+1} = X_n + X_{n-1}$$

Applying our induction hypothesis twice, we get

$$X_{n+1} = (b \cdot f_n + a \cdot f_{n-1}) + (b \cdot f_{n-1} + a \cdot f_{n-2})$$

$$= b \cdot (f_n + f_{n-1}) + a \cdot (f_{n-1} + f_{n-2})$$

$$= b \cdot f_{n+1} + a \cdot f_n$$

The last line comes from the Fibonacci property.

Please note that, in the inductive step when $n + 1 = 3$, we used the fact that (*) holds for both $n = 2$ and $n = 1$. Therefore we needed to prove two consecutive base cases (the $X_2$ case as well as the $X_1$ case).

**Exercise 2** (10 points) How many different arrangements can be formed using the letters Manhattan?

**Solution:** In Manhattan there are 3 $a$, 2 $n$, 2 $t$, 1 $m$ and 1 $h$ for a total of 9 letters. Theorem 22.9 from the notes gives the number of different arrangement via th following formula. If you don’t recall it, you can still find it saying there are $9!$ arrangements if all the letters were different, and you count a same arrangement $n_i!$ times when letter $i$ appears $n_i$ times.

$$n = \frac{9!}{3! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = 15120$$

**Exercise 3** (10 points) How many positive integers with five digits or less have neither their first digit equal to 3 nor their last digit equal to 5?

**Solution:** We are going to count the relevant number of integers by considering successively integers with exactly one digit, then exactly 2 digits, etc up to 5 digits, each time with the formula: total number of integers minus those which are disqualified.
1. Between 0 and 9, there are \(10 - 1 - 1 - 1 = 7\) relevant numbers. (0 is not positive, 3 and 5 are not allowed).

2. Between 10 and 99, there are \(90 - 10 - 9 + 1 = 72\) relevant numbers. There are 90 numbers between 10 and 99. Among them 10 begin by a 3 (30, 31, 32, \ldots, 39). Among them 9 end by a 5 (15, 25, 35, 45, etc.). And we counted twice 35, so we add 1. Another way to count is the following: 8 possibilities for the first digit (shouldn’t be 0 nor 3), and 9 for the last one, that is \(8 \times 9 = 72\).

3. Between 100 and 999, there are \(900 - 100 - 90 + 10 = 720\). Same reasoning, there are 900 numbers between 100 and 999, among them 100 start by a 3 and 99 end by a 5. If you don’t see the 90, it comes from 100 (all number between 0 and 999 ending by a 5, minus the 10 ones which only have 1 or 2 digits. Here again we have counted 305, 315, 325, etc. twice. Or, 8 possibilities for the first digit (not 0 nor 3), 10 for the one in the middle, and 9 for the last one (not 5), \(8 \times 10 \times 9 = 720\).

4. Between 1000 and 9999, there are \(9,000 - 1000 - 900 + 100 = 8 \times 10 \times 10 \times 9 = 7,200\).

5. Between 10000 and 99999, there are \(90,000 - 10,000 - 9,000 + 1,000 = 8 \times 10 \times 10 \times 10 \times 9 = 72,000\).

There are \(N\) of those numbers:

\[
N = 7 + 72 + 720 + 7,200 + 72,000 = 79,999
\]

Exercise 4 (10 points) Using Pascal’s Triangle expand \((2x - y)^7\). Draw the corresponding portion of the Triangle. Feel free not to simplify the coefficients.

Solution:

\[
(2x - y)^7 = \sum_{i=0}^{7} C_i^7 (-1)^{7-i} 2^i x^i y^{7-i}
\]

\begin{align*}
&= 2^7 x^7 - 7 \cdot 2^6 x^6 y + 21 \cdot 2^5 x^5 y^2 - 35 \cdot 2^4 x^4 y^3 \\
&\quad + 35 \cdot 2^3 x^3 y^4 - 21 \cdot 2^2 x^2 y^5 + 7 \cdot 2xy^6 - y^7 \\
&= 128x^7 - 448x^6 y + 672x^5 y^2 - 560x^4 y^3 \\
&\quad + 280x^3 y^4 - 84x^2 y^5 + 14xy^6 - y^7
\end{align*}

Pascal’s Triangle expanded to level 7

Exercise 5 (10 points) How many positive integers are there less than 10000 such that the sum of their decimal digits is 12?

Solution: We have

\[
x_1 + x_2 + x_3 + x_4 = 12
\]
Where the $x_i$’s are nonnegative integer; as usual for this type of problem $x_i$ represents the $i^{th}$ digit.

The number we are looking for is the number of solutions to (1) subject to the constraint (C).

\[ x_1 \leq 9 \text{ and } x_2 \leq 9 \text{ and } x_3 \leq 9 \text{ and } x_4 \leq 9 \quad (C) \]

The number of solutions to (1) subject to (C) is the number of solutions without constraints minus the number of solutions with the complement ($D_1$) of the constraint (C).

\[ x_1 \geq 10 \text{ or } x_2 \geq 10 \text{ or } x_3 \geq 10 \text{ or } x_4 \geq 10 \quad (D_1) \]

If we look at the complement of the constraint: we get a disjunction as the constraint.

Now, the key observation is that the four disjuncts are pairwise disjoint, since if $x_i$ and $x_j$ ($i \neq j$) are both larger or equal to 10, then the sum clearly exceeds 12. So the number of solutions to (1) subject to ($D_1$) is simply four times the number of solutions to (1) subject to ($D_2$).

\[ x_1 \geq 10 \quad (D_2) \]

The standard techniques (”stars and bars”) yield:

- (1) unconstrained: \( \binom{12}{4} \cdot \binom{15}{1} = 455 \) solutions
- (1) subject to ($D_2$): \( \binom{2}{4} \cdot \binom{4}{1} = 10 \) solutions

Subtracting, we get the number of solutions of (1) subject to (C): \( 455 - 4 \times 10 = 415 \) solutions.

**Exercise 6** (10 points) The deck of cards contains 52 cards. There are 13 different kinds of cards: 2,3,4,5,6,7,8,9,10,J,D,K,A. There are also four suits: spades, clubs, hearts, and diamonds, each containing 13 cards, with one card of each kind in a suit. What is the probability that a given poker hand of five cards is a royal flush (A,K,D,J,10 of the same suit)?

**Solution:** There are only four ways of getting a royal flush: (A ♠ K ♠ D ♠ J ♠ 10 ♠), (A ♥ K ♥ D ♥ J ♥ 10 ♥), (A ♦ K ♦ D ♦ J ♦ 10 ♦), (A ♣ K ♣ D ♣ J ♣ 10 ♣).

There is a total of \( \binom{5}{5} \) possible hands. Therefore the probability of getting a royal flush is \( P = \frac{4}{\binom{5}{5}} = \frac{4}{1} = 1.54 \times 10^{-6} \).

**Note:** Especially in probabilities, it is very easy to get things wrong, you should *always* explain what you do, a bunch of numbers doesn’t prove anything, and nobody has a clue of what you are doing. I gave more credits to wrong answers with explanations than to the presumably same wrong answers without explanations.

**Exercise 7** (10 points) A fair coin is tossed five times. What is the probability of getting exactly four heads, given that at least one of the tosses is heads?

**Solution:** The number of relevant possibilities is \( \binom{5}{4} = 5 \), and the total number of possibilities is all those which have at least one heads, that is \( 2^5 \) minus the number of possibilities with no heads at all, which is 1. Therefore \( P = \frac{5}{2^5 - 1} = \frac{5}{31} \approx 0.16 \).
Exercise 8 (20 points) Some tribe values boys so much that each of their families keeps making kids until they get a boy (after which they relax and make no more kids). On the other hand, no family can afford having more than five kids. So if the first five babies in a family are girls, the family stops making children anyway. Assuming that a boy and a girl are equally likely, consider two random variables $X$ - “the number of boys in a family”, $Y$ - “the number of girls in a family”.

a) Find the expected values $E(X)$ and $E(Y)$ and compare them.

b) Are $X$ and $Y$ independent?

c) Find the expected value $E(X + Y)$.

Solution:

A family will always raise exactly one boy, except if they get 5 girls, in which case they won’t get any boy. So $P(\text{one boy}) = 1 - \frac{1}{2^5} = \frac{31}{32}$ and $E(\text{number of boys}) = E(X) = \frac{31}{32} \approx 0.97$ boys.

$$P(\text{no girl}) = P(Y = 0) = \frac{1}{2} \quad \text{A boy as their first kid}$$

$$P(\text{one girl}) = P(Y = 1) = \frac{1}{2} \cdot \frac{1}{2} \quad \text{A girl and then a boy}$$

$$P(\text{two girls}) = P(Y = 2) = \left(\frac{1}{2}\right)^3 \quad \text{A girl, a girl again, and then a boy}$$

$$P(\text{three girls}) = P(Y = 3) = \left(\frac{1}{2}\right)^4 \quad \text{A girl three times, and then a boy}$$

$$P(\text{four girls}) = P(Y = 4) = \left(\frac{1}{2}\right)^5 \quad \text{A girl four times, and then a boy}$$

$$P(\text{five girls}) = P(Y = 5) = \left(\frac{1}{2}\right)^5 \quad \text{Five girls, and then they stop}$$

$$E(\text{number of girls}) = E(Y) = \sum_{i=0}^{5} i \cdot P(Y = i) = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{32} = \frac{8 + 8 + 6 + 4 + 5}{32} = \frac{31}{32} \approx 0.97 \text{ girls}$$

a) The expected number of boys is equal to the expected number of girls.

b) $X$ and $Y$ are certainly not independent since $(Y = 5) \Rightarrow (X = 0)$.

c) $E(X + Y)$ is always $E(X) + E(Y)$ even when $X$ and $Y$ are not independent.

$$E(X + Y) = 2 \cdot \frac{31}{32} = \frac{31}{16} \approx 1.94 \text{ children}$$

You can also consider $Z$ the random variable number of children $Z = X + Y$, compute the $P(Z = i) = \frac{1}{2^i} = P(Y = i - 1)$ for $i \in \{1, 2, 3, 4\}$, $P(Z = 5) = \frac{1}{16} = P(Y = 4) = P(Y = 5)$. And then $E(Z) = \sum_{i=1}^{5} i \cdot P(Z = i) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{16} = \frac{8 + 8 + 6 + 4 + 5}{16} = \frac{31}{16}$, but this is a waste of time.