Dijkstra’s Algorithm: Correctness

Why is this right? Getting the right loop invariant is hard.

- What does $d(v_i)$ correspond to if $v_i \not\in U$?

Loop invariant. For all $k \geq 0$, after the $k$th iteration of the loop:

- $U$ consists of $v_0$ and the $k$ vertices closest to $v_0$
- $u$ is the $k$th closest vertex to $v_0$
- if $v \in U$, then $d(v)$ is the length of the shortest path from $v_0$ to $v$.
- if $v \not\in U$, then $d(v)$ is the length of the shortest path from $v_0$ to $v$ that has only vertices in $U$ as intermediate points.
- if there is no such path, then $d(v) = \infty$
- $u$ is the most recently added vertex.

Once we get the right loop invariant, it’s not hard to prove that it is maintained ... by induction (of course):

**Basis:** $P(0)$ is obvious.

**Inductive step:** ...

Theorem: A connected (multi)graph has an Eulerian cycle iff each vertex has even degree.

**Proof:** The necessity is clear: In the Eulerian cycle, there must be an even number of edges that start or end with any vertex.

To see the condition is sufficient, we provide an algorithm for finding an Eulerian circuit in $G(V,E)$.

First step: Follow your nose to construct a cycle.

Second step: Remove the edges in the cycle from $G$.

Let $H$ be the subgraph that remains.

- every vertex in $H$ has even degree
- $H$ may not be connected; let $H_1, \ldots, H_h$ be its connected components.

Third step: Apply the algorithm recursively to $H_1, \ldots, H_h$, and then splice the pieces together.

Eulerian Paths

Recall that $G(V,E)$ has an Eulerian path if it has a path that goes through every edge exactly once. It has an Eulerian cycle (or Eulerian circuit) if it has an Eulerian path that starts and ends at the same vertex.

How can we tell if a graph has an Eulerian path/circuit?

What’s a necessary condition for a graph to have an Eulerian circuit?

Count the edges going into and out of each vertex:

- Each vertex must have even degree!

This condition turns out to be sufficient too.

Finding cycles

First, find an algorithm for finding a cycle:

**Input:** $G(V,E)$ [a list of vertices and edges]

**procedure** Pathgrow($V,E,v$)

- $[v$ is first vertex in cycle$]$ $P \leftarrow ()$ [P is sequence of edges on cycle$]$ $w \leftarrow v$ [w is last vertex in P$]$

repeat until $I(w) - P = 0$

- $[I(w)$ is the set of edges incident on $w$]

Pick $e \in I(w) - P$

$w \leftarrow$ other end of $e$

$P \leftarrow P \cdot e$ [append $e$ to $P$]

endrepeat

return $P$

endpro

**Claim:** If every vertex in $V$ has even degree, then $P$ will be a cycle

- Loop invariant: In the graph $G(V,E - P)$, if the first vertex ($v$) and last vertex ($w$) in $P$ are different, they have odd degree; all the other vertices have even degree.
Finding Eulerian Paths

**Input:** \( G(V, E) \) [a list of vertices and edges]

**Algorithm ECycle:**

```plaintext
procedure Euler(V', E', v')
  Pathgrow(V', E', v')
  if \( P \) is not Eulerian,
     delete the edges in \( P \) from \( E' \);
     let \( G_1(V_1, E_1), \ldots, G_n(V_n, E_n) \) be
     the resulting connected components
     let \( v_i \) be a vertex in \( V_i \)
     for \( i = 1 \) to \( n \)
       Euler(V_i, E_i, v_i)
       Attach \( C \) to \( P \) at \( v_i \)
   endfor
   \( C \leftarrow P \)
   return \( C \)
endpro
```

\( v \leftarrow \) any vertex in \( V \)
Euler(V, E, v)

---

Hamiltonian Paths

Recall that \( G(V, E) \) has a Hamiltonian path if it has a path that goes through every vertex exactly once. It has a Hamiltonian cycle (or Hamiltonian circuit) if it has a Hamiltonian path that starts and ends at the same vertex.

There is no known easy characterization or algorithm for checking if a graph has a Hamiltonian cycle/path.

Which of these graphs have a Hamiltonian cycle?

---

Searching Graphs

Suppose we want to process data associated with the vertices of a graph. This means we need a systematic way of searching the graph, so that we don’t miss any vertices.

There are two standard methods.

- **Breadth-first search**
- **Depth-first search**

It’s best to think of these on a tree:

Breadth-first search would visit the nodes in the following order:

\[ 1, 2, 3, \ldots, 10 \]

Depth-first search would visit the nodes in the following order:

\[ 1, 2, 4, 5, 7, 8, 11, 3, 6, 9, 10 \]
**Breadth-First Search**

**Input** \( G(V, E) \) [a connected graph]  
\( v \) [start vertex]

**Algorithm Breadth-First Search**

visit \( v \)

\[ V' \leftarrow \{v\} \]  
\( V' \) is the vertices already visited

Put \( v \) on \( Q \)  
\( Q \) is a queue

repeat while \( Q \neq \emptyset \)

\( u \leftarrow \text{head}(Q) \)  
\( \text{head}(Q) \) is the first item on \( Q \)

for \( w \in A(u) \)  
\( A(u) = \{w \mid \{u, w\} \in E\} \)

if \( w \notin V' \)
then
visit \( w \)

Put \( w \) on \( Q \)

\[ V' \leftarrow V' \cup \{w\} \]
endif
endfor

Delete \( u \) from \( Q \)

If all edges have equal length, we can extend this algorithm to find the shortest path length from \( v \) to any other vertex:

- **Store the path length with each node when you add it.**
- **Length**\( (v) = 0.\)
- **Length**\( (w) = \text{Length}(u) + 1 \)

**Depth-First Search**

**Input** \( G(V, E) \) [a connected graph]  
\( v \) [start vertex]

**Algorithm Depth-First Search**

visit \( v \)

\[ V' \leftarrow \{v\} \]  
\( V' \) is the vertices already visited

Put \( v \) on \( S \)  
\( S \) is a stack

\( u \leftarrow v \)

repeat while \( S \neq \emptyset \)

if \( A(u) - V' \neq \emptyset \)
then
Choose \( w \in A(u) - V' \)

visit \( w \)

\[ V' = V' \cup \{w\} \]

Put \( w \) on stack

\( u \leftarrow w \)
else
\( u \leftarrow \text{top}(S) \)  
[Pop the stack]
endif
endrepeat