Inductive Definitions

Example: Define $\sum_{k=1}^{n} a_k$ inductively (i.e., by induction on $n$):

- $\sum_{k=1}^{n} a_k = a_1$
- $\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}$

The inductive definition avoids the use of $\cdots$ and thus is less ambiguous.

Example: An inductive definition of $n!$:

- $1! = 1$
- $(n+1)! = (n+1)n!$

Could even start with $0! = 1$.

Theorem: $P = P'$. (The two approaches define the same set.)

Proof: Show $P \subseteq P'$ and $P' \subseteq P$.

To see that $P \subseteq P'$, it suffices to show that

(a) $P'$ contains $a, b, c, d, aa, bb, cc, dd$

(b) if $x$ is in $P'$, then so is $axa, bxb, cxc, and dxd$

(since $P$ is the least set with these properties).

Clearly $P_1 \cup P_2$ satisfies (1), so $P'$ does. And if $x \in P'$, then $x \in P_n$ for some $n$, in which case $axa, bxb, cxc, and dxd$ are all in $P_{n+2}$ and hence in $P'$. Thus, $P \subseteq P'$.

To see that $P' \subseteq P$, we prove by strong induction that $P_n \subseteq P$ for all $n$. Let $P(n)$ be the statement that $P_n \subseteq P$.

Basis: $P_1, P_2 \subseteq P$: Obvious.

Suppose $P_1, \ldots, P_n \subseteq P$. If $n \geq 2$, the fact that $P_{n+1} \subseteq P$ follows immediately from (b). (Actually, all we need is the fact that $P_{n-1} \subseteq P$, which follows from the (strong) induction hypothesis.)

Thus, $P' = \bigcup_n P_n \subseteq P$.

Inductive Definitions of Sets

A palindrome is an expression that reads the same backwards and forwards:

- Madam I’m Adam
- Able was I ere I saw Elba

What is the set of palindromes over $\{a, b, c, d\}$? Two approaches:

1. The smallest set $P$ such that

(a) $P$ contains $a, b, c, d, aa, bb, cc, dd$

(b) if $x$ is in $P$, then so is $axa, bxb, cxc, and dxd$

“Smallest” is not in terms of cardinality.

- $P$ is guaranteed to be infinite

“Smallest” is in terms of the subset relation.

Here’s a set that satisfies (a) and (b) and isn’t the smallest:

Define $Q_n$ inductively:

- $Q_1 = \{a, b, c, d\}$
- $Q_2 = \{aa, bb, cc, dd, ab\}$
- $Q_{n+1} = \{axa, bxb, cxc, dxd | x \in Q_{n-1}\}, n \geq 2$

Let $Q = \bigcup_n Q_n$.

It’s easy to see that $Q$ satisfies (a) and (b), but it isn’t the smallest set to do so.
**Just a Reminder**

(from your friendly sponsor)

What’s (usually) a key step in proving a property of an algorithm:

Find a loop invariant!

- State clearly what the invariant is
- Prove that it holds (often by induction, since the invariant says “On the $n$th iteration of the loop, property $P(n)$ holds”)

**Graphs and Trees**

Graphs and trees come up everywhere. We saw an example in Chapter 0 of a precedence graph. Here’s another example of where graphs come in handy:

A farmer is bringing a wolf, a cabbage, and a goat to market. They need to cross a river in a boat which can accommodate only two things, including the farmer. Moreover:

- the farmer can’t leave the wolf alone with the goat
- the farmer can’t leave the goat alone with the cabbage

How should he cross the river?

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**Other Examples**

**Niche graphs (Ecology):**

- The vertices are species
- Two vertices are connected by an edge if they compete (use the same food resources, etc.)

Niche graphs give a visual representation of competitiveness.

**Influence Graphs**

- The vertices are people
- There is an edge from $a$ to $b$ if $a$ influences $b$

Influence graphs give a visual representation of power structure.

There are lots of other examples in all fields …
Terminology and Notation

A graph $G$ is a pair $(V, E)$, where $V$ is a set of vertices or nodes and $E$ is a set of edges or branches: an edge is a set $\{v, v'\}$ of two not necessarily distinct vertices (i.e., $v, v' \in V$).

- We sometimes write $G(V, E)$ instead of $G$.
- If $V = \emptyset$, then $E = \emptyset$, and $G$ is called the null graph.

We usually represent a graph pictorially.

- A vertex with no edges incident to it is said to be isolated.
- If $\{v\} \in E$ (the book writes $\{v, v\}$), then there is a loop at $v$.
- $G'(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Directed Graphs

Note that $\{v, u\}$ and $\{u, v\}$ represent the same edge. In a directed graph (digraph), the order matters. We denote an edge as $(v, v')$ rather than $\{v, v'\}$. We can identify an undirected graph with the directed graph that has edges $(v, v')$ and $(v', v)$ for every edge $\{v, v'\}$ in the undirected graph.

Two vertices $v$ and $v'$ are adjacent if there is an edge between them, i.e., $(v, v') \in E$ in the undirected case, $(v, v') \in E$ or $(v', v) \in E$ in the directed case.

Representing Relations Graphically

Given a relation $R$ on $S \times T$, we can represent it by the directed graph $G(V, E)$, where

- $V = S \cup T$ and
- $E = \{(s, t) : (s, t) \in R\}$

Example: Represent the $<$ relation on $\{1, 2, 3, 4\}$ graphically.

How does the graphical representation show that a graph is

- reflexive?
- symmetric?
- transitive?

Multigraphs

In a multigraph, there may be several edges between two vertices.

- There may be several roads between two towns.
- There may be several transformations that can change you from one configuration to another.
  - This is particularly important in graphs where edges are labeled.

Formally, a multigraph $G(V, E)$ consists of a set $V$ of vertices and a multiset $E$ of edges.

- The same edge can be in more than once.

In this course, all graphs are simple graphs (not multigraphs) unless explicitly stated otherwise.

- Most of the results generalize to multigraphs.
Degree

In a directed graph $G(V,E)$, the \textit{indegree} of a vertex $v$ is the number of edges coming into it

• $\text{indegree}(v) = |\{v' : (v',v) \in E\}|$

The \textit{outdegree} of $v$ is the number of edges going out of it:

• $\text{outdegree}(v) = |\{v' : (v,v') \in E\}|$

The degree of $v$, denoted $\deg(v)$, is the sum of the indegree and outdegree.

For an undirected graph, it doesn’t make sense to talk about indegree and outdegree. The degree of a vertex is the sum of the edges incident to the vertex, except that we double-count all self-loops.

• Why? Because things work out better that way

Handshaking Theorem

Theorem: The number of people who shake hands with an odd number of people at a party must be even.

Proof: Construct a graph, whose vertices are people at the party, with an edge between two people if they shake hands. The number of people person $p$ shakes hands with is $\deg(p)$. Split the set of all people at the party into two subsets:

- $A =$ those that shake hands with an even number of people
- $B =$ those that shake hands with an odd number of people

$$\sum_{p} \deg(p) = \sum_{p \in A} \deg(p) + \sum_{p \in B} \deg(p)$$

- We know that $\sum_{p} \deg(p) = 2|E|$ is even.
- $\sum_{p \in A} \deg(p)$ is even, because for each $p \in A$, $\deg(p)$ is even.
- Therefore, $\sum_{p \in B} \deg(p)$ is even.

Therefore $|B|$ is even (because for each $p \in B$, $\deg(p)$ is odd, and if $|B|$ were odd, then $\sum_{p \in B} \deg(p)$ would be odd).
**Paths**

Given a graph $G(V, E)$.
- A *path* in $G$ is a sequence of vertices $(v_0, \ldots, v_n)$ such that $\{v_i, v_{i+1}\} \in E$ ($(v_i, v_{i+1})$ in the directed case).
- If $v_0 = v_n$, the path is a *cycle*
- An *Eulerian* path/cycle is a path/cycle that traverses every every edge in $E$ exactly once
- A *Hamiltonian* path/cycle is a path/cycle that passes through each vertex in $V$ exactly once.
- A graph with no cycles is said to be *acyclic*

**Connectivity**

- An undirected graph is *connected* if there is for all vertices $u, v$, $(u \neq v)$ there is a path from $u$ to $v$.
- A digraph is *strongly connected* if for all vertices $u, v$, ($u \neq v$) there is a path from $u$ to $v$ and from $v$ to $u$.
- If a digraph is connected but not strongly connected, it is *weakly connected*.
- A *connected component* of $G$ is a connected subgraph $G'$ which is not the subgraph of any other connected subgraph of $G$.

**Example:** We want the graph describing the interconnection network in a parallel computer:
- the vertices are processors
- there is an edge between two nodes if there is a direct link between them.
  - if links are one-way links, then the graph is directed

We typically want this graph to be connected.

**Trees**

A *tree* is a digraph such that
(a) with edge directions removed, it is connected and acyclic
(b) every vertex but one, the root, has indegree 1
(c) the root has indegree 0

Trees come up everywhere:
- when analyzing games
- representing family relationships

**Bipartite Graphs**

A graph $G(V,E)$ is *bipartite* if we can partition $V$ into disjoint sets $V_1$ and $V_2$ such that all the edges in $E$ joins a vertex in $V_1$ to one in $V_2$.

**Example:** Suppose we want to represent the “is or has been married to” relation on people. Can partition the set $V$ of people into males ($V_1$) and females ($V_2$). Edges join two people who are or have been married.
Complete Graphs and Cliques

- An undirected graph $G(V, E)$ is complete if it has no loops and for all vertices $u, v$ ($u \neq v$), $\{u, v\} \in E$.
  - How many edges are there in a complete graph with $n$ vertices?

A complete subgraph of a graph is called a clique

- The clique number of $G$ is the size of the largest clique in $G$. 