Independence

Intuitively, events $A$ and $B$ are independent if they have no effect on each other.

This means that observing $A$ should have no effect on the likelihood we ascribe to $B$, and similarly, observing $B$ should have no effect on the likelihood we ascribe to $A$.

Thus, if $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$ and $A$ is independent of $B$, we would expect

$$
\Pr(B|A) = \Pr(B) \text{ and } \Pr(A|B) = \Pr(A).
$$

**The Second-Child Problem**

Suppose that any child is equally likely to be male or female, and that the sex of any one child is independent of the sex of the other. You have an acquaintance and you know he has two children, but you don’t know their sexes. Consider the following four cases:

1. You visit the acquaintance, and a boy walks into the room. The acquaintance says “That’s my older child.”

2. You visit the acquaintance, and a boy walks into the room. The acquaintance says “That’s one of my children.”

3. The acquaintance lives in a culture, where male children are always introduced first, in descending order of age, and then females are introduced. You visit the acquaintance, who says “Let me introduce you to my children.” Then he calls “John [a boy], come here!”

4. You go to a parent-teacher meeting. The principal asks everyone who has at least one son to raise their hands. Your acquaintance does so.

In each case, what is the probability that the acquaintance’s second child is a boy?

- The problem is to get the right sample space

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**Probability Trees**

Suppose that the probability of rain tomorrow is .7. If it rains, then the probability that the game will be cancelled is .8; if it doesn’t min, then the probability that it will be cancelled is .1. What is the probability that the game will be played?

The situation can be described by a tree:

Similar trees can be used to describe

- Sequential decisions
- Randomized algorithms
Bayes’ Theorem

**Bayes Theorem:** Let $A_1, \ldots, A_n$ be mutually exclusive and exhaustive events in a sample space $S$.

- That means $A_1 \cup \ldots \cup A_n = S$, and the $A_i$’s are pairwise disjoint: $A_i \cap A_j = \emptyset$ if $i \neq j$.

Let $B$ be any other event in $S$. Then

$$\Pr(A_i | B) = \frac{\Pr(A_i) \Pr(B | A_i)}{\sum_{j=1}^{n} \Pr(A_j) \Pr(B | A_j)}.$$  

**Proof:**

$$B = B \cap S = B \cap (\bigcup_{j=1}^{n} A_j) = \bigcup_{j=1}^{n} (B \cap A_j),$$

therefore, $\Pr(B) = \sum_{j=1}^{n} \Pr(B \cap A_j)$.

Next, observe that $\Pr(B | A_i) = \Pr(A_i \cap B) / \Pr(A_i)$. Thus,

$$\Pr(A_i \cap B) = \Pr(B | A_i) \Pr(A_i).$$

Therefore,

$$\Pr(A_i | B) = \frac{\Pr(A_i \cap B)}{\Pr(B)}
= \frac{\Pr(B | A_i) \Pr(A_i)}{\sum_{j=1}^{n} \Pr(B | A_j) \Pr(A_j)}.$$

Using Bayes’ Theorem, we have:

$$\Pr(A_2 | B) = \frac{\Pr(B | A_2) \Pr(A_2)}{\Pr(B | A_1) \Pr(A_1) + \Pr(B | A_2) \Pr(A_2) + \Pr(B | A_3) \Pr(A_3)}$$

$$= \frac{(.45 \times .6) + (.65 \times .3) + (.55 \times .1)}{.99}$$

$$\approx .398$$

**Example**

In a certain county, 60% of registered voters are Republicans, 30% are Democrats, and 10% are Independents. 40% of Republicans oppose increased military spending, while 65% of the Democrats and 55% of the Independents oppose it. A registered voter writes a letter to the county paper, arguing against increased military spending. What is the probability that this voter is a Democrat?

$S = \{\text{registered voters}\}$

$A_1 = \{\text{registered Republicans}\}$

$A_2 = \{\text{registered Democrats}\}$

$A_3 = \{\text{registered independents}\}$

$B = \{\text{voters who oppose increased military spending}\}$

We want to know $\Pr(A_2 | B)$.

We have

$$\Pr(A_1) = .6 \quad \Pr(A_2) = .3 \quad \Pr(A_3) = .1$$

$$\Pr(B | A_1) = .4 \quad \Pr(B | A_2) = .65 \quad \Pr(B | A_3) = .55$$

**AIDS**

Suppose we have a test that is 99% effective against AIDS. Suppose we also know that .3% of the population has AIDS. What is the probability that you have AIDS if you test positive?

$S = \{\text{all people}\}$ (in North America??)

$A_1 = \{\text{people with AIDS}\}$

$A_2 = \{\text{people who don’t have AIDS}\}$ ($A_2 = \overline{A_1}$)

$B = \{\text{people who test positive}\}$

$$\Pr(A_1) = .003 \quad \Pr(A_2) = .997$$

Since the test is 99% effective:

$$\Pr(B | A_1) = .99 \quad \Pr(B | A_2) = .01$$

Using Bayes’ Theorem again:

$$\Pr(A_1 | B) = \frac{.99 \times .003}{.99 \times .003 + .01 \times .997}$$

$$\approx \frac{.00297}{.00397}$$

$$\approx .75$$
Averaging and Expectation

Suppose you toss a coin that’s biased towards heads (Pr(heads) = 2/3) twice. How many heads do you expect to get?

• In mathematics-speak:
  What’s the expected number of heads?
What about if you toss the coin $k$ times?
What’s the average weight of the people in this classroom?
• That’s easy: add the weights and divide by the number of people in the class.
But what about if I tell you I’m going to toss a coin to determine which person in the class I’m going to choose; if it lands heads, I’ll choose someone at random from the first aisle, and otherwise I’ll choose someone at random from the last aisle.
• What’s the expected weight?
Averaging makes sense if you use an equiprobable distribution; in general, we need to talk about expectation.

Random Variables

To deal with expectation, we formally associate with every element of a sample space a real number.

Definition: A random variable on sample space $S$ is a function from $S$ to the real numbers.

Example: Suppose we toss a biased coin (Pr($h$) = 2/3) twice. The sample space is:

• $hh$ - Probability 4/9
• $ht$ - Probability 2/9
• $th$ - Probability 2/9
• $tt$ - Probability 1/9

If we’re interested in the number of heads, we would consider a random variable $#H$ that counts the number of heads in each sequence:

$#H(hh) = 2; \ #H(ht) = #H(th) = 1; \ #H(tt) = 0$

Example: If we’re interested in weights of people in the class, the sample space is people in the class, and we could have a random variable that associates with each person his or her weight.