Answer to 4.3-8: (2 points, DC) Label the runners 1 through 5. The question then asks in how many ways we can arrange these numbers. There are 5 ways to choose a first, then 4 ways to choose a second, then 3 ways to choose a third, then 2 ways to choose a fourth, and then one way to choose the last. So the answer is 5! □

Answer to 4.3-16: (4 points, MH)
(b) There are 5 1’s, and the other 5 bits are 0’s. The question amounts to asking how many ways we can choose 5 positions out of 10 to be assigned the bit 1. The answer is \(C(10, 5) = 252\).
(d) There is 1 \(C(10, 0)\) string containing zero 1’s. There are 10 \(C(10, 1)\) strings containing exactly one 1. And there are 45 \(C(10, 2)\) strings containing exactly two 1’s. There are \(2^{10} = 1024\) strings altogether. So the number of strings having at least three 1’s is \(1024 - 45 - 10 - 1 = 968\). □

Answer to 4.3-24b: (3 points, DW) These strings must have at least one a and one b. There are \(C(6, 2) = 15\) ways to choose the two positions of the two letters. Since there are two possibilities for each choice (a before b, and b before a), there are 30 ways to put the letters a and b in a six-letter string. There are 4 positions left open in the string and we can fill these 4 with any of the 26 letters of the alphabet. So the answer is \(30 \times 26^4 = 13,709,280\). □

Answer to 4.3-32: (2 points, FW) There are 6! ways to seat them at the table. But because of symmetry, we have overcounted by a factor of 6. So the answer is 6!/6 = 5!. □

Answer to 4.4-8: (5 points, HC) The sample space \(S\) is the set of all 5-card poker hands. The size (cardinality) of \(S\) is the number of distinct ways 5 cards can be chosen from 52 (the size of a deck). So \(|S| = C(52, 5)\). The event \(E\) is the subset of \(S\) in which each element (i.e., each hand of 5 cards) contains the ace of hearts. To calculate \(|E|\), note that once we know that a hand contains the ace of hearts, there remain \(C(51, 4)\) ways to complete that hand: we must choose four more cards from 51 remaining cards. So \(|E| = C(51, 4)\). The probability is therefore

\[
p(E) = \frac{|E|}{|S|} = \frac{C(51, 4)}{C(52, 5)} = \frac{51! \cdot 5! \cdot 47!}{52! \cdot 4! \cdot 47!} = \frac{51!}{5!} = \frac{52!}{4!} = \frac{9.6%}{52} \]

□
**Answer to 4.4-14:** (4 points, JA) The sample space $S$ is the set of all five-card hands, so

$$|S| = \binom{52}{5} = \frac{52!}{47! \cdot 5!}$$

since there are $\binom{52}{5}$ ways to choose 5 cards from 52. The event space $E$ is the set of all hands containing five cards of different kinds. There are 52 ways to choose the first card. Since there are four suits, though, there remain only 48 ways of choosing the second card. Then there are only 44 ways of choosing the third; then 40 of choosing the fourth; and finally, 36 of choosing the last. So there are $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$ different hands of five different kinds, taking into account the order in which we choose them. Since we are not interested in the order (for example, we consider $(\text{Jack}, 3, 4, 9, \text{Ace})$ to be the same hand as $(3, 9, \text{Jack}, \text{Ace}, 4)$), we must divide by the number of different permutations of five elements, which is $5!$. We conclude that

$$|E| = \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5!}$$

$$p(E) = \frac{|E|}{|S|} = \frac{5! \cdot 52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{5! \cdot 52!}$$

$$= \frac{44 \cdot 40 \cdot 36}{51 \cdot 50 \cdot 49} 
\approx 51\%.$$ 

So a hand of distinct kinds is quite likely. \hfill \Box

**Answer to 4.4-24:** (3 points, JA)

(a) The sample space $S$ is the set of all sets of 6 distinct integers between 1 and 36. There are $\binom{30}{6} = 593,775$ different choices, so this is $|S|$. The event is a single element of $S$, so the probability is $1/|S| = 1/593,775 = 0.00017\%$.

(d) Same as in (a): $|S| = \binom{48}{6} \Rightarrow p = 1/|S| = 1/\binom{48}{6} = 0.000008\%$. \hfill \Box

**Answer to 4.4-32:** (5 points, MH) We calculate the probability for both cases.

- *Two dice.* The sample space $S$ is the set of all rolls. There are six possibilities for each die, so $|S| = 6^2 = 36$. The event $E$ is the set of all rolls adding up to 8. In how many ways can we do this? Answer: a 2 and a 6; a 3 and a 5; a 4 and a 4; a 5 and a 3; and finally, a 6 and a 2. In other words, there are 5 ways, so $|E| = 5$. By the way, if you want to define $S$ and $E$ formally, here’s how to do it:

$$R = \{1, 2, 3, 4, 5, 6\}$$

$$S = \{(x, y) \mid x, y \in R\}$$

$$E = \{(x, y) \in S \mid x + y = 8\}$$

Anyway, we have $p(E) = |E|/|S| = 5/36 \approx 13.9\%$. 

Three dice. Same as above, but for three dice:

\[ S = \{(x, y, z) \mid x, y, z \in R\} \]
\[ E = \{(x, y, z) \in S \mid x + y + z = 8\} \]

By a brute-force count, you can show that there are 21 ways to solve the equation in the definition of \( E \), so \(|E| = 21\). (Soon you will learn a more elegant and powerful way to determine the number of ways in which integer equation can be solved). Since \(|S| = 6^3 = 216\), the probability is \( p(E) = 21/216 \approx 9.7\% \).

Conclusion: you’re likelier to roll 8 with 2 dice than with 3.