Handed out: Tues, Apr. 24 on the web.
Here is the analysis of the local truncation error of RK2 that I did not have time to finish in lecture. Recall that RK2 is given by the following sequence of formulas, which assume $y_n$ is already computed.

\[
\begin{align*}
k_1 &= f(t_n, y_n), \\
k_2 &= f(t_n + h, y_n + hk_1), \\
y_{n+1} &= y_n + (h/2)(k_1 + k_2). \\
\end{align*}
\]

To analyze the order of this method, we first substitute the exact solution into the finite difference formula, inserting a remainder term $R$, and obtaining:

\[
\begin{align*}
k_1 &= f(t_n, y(t_n)), \\
k_2 &= f(t_n + h, y(t_n) + hk_1), \\
y(t_{n+1}) &= y(t_n) + (h/2)(k_1 + k_2) + R. \\
\end{align*}
\]

The goal is now to find a formula for $R$ as a Taylor series. First, we expand $k_2$ via a Taylor series:

\[
k_2 = f(t_n, y(t_n)) + h(\partial / \partial t)f(t_n, y(t_n)) + hk_1(\partial / \partial y)f(t_n, y(t_n)) + O(h^2) = f(t_n, y(t_n)) + h(\partial / \partial t)f(t_n, y(t_n)) + hf(t_n, y(t_n))(\partial / \partial y)f(t_n, y(t_n)) + O(h^2).
\]

Substitute these formulas for $k_1, k_2$ into (1) to obtain

\[
y(t_{n+1}) = y(t_n) + (h/2)[f(t_n, y(t_n)) + f(t_n, y(t_n)) + h(\partial / \partial t)f(t_n, y(t_n)) + hf(t_n, y(t_n))(\partial / \partial y)f(t_n, y(t_n)) + O(h^2)] + R
\]

\[= y(t_n) + h(f(t_n, y(t_n)) + (h^2/2)(\partial / \partial t)f(t_n, y(t_n)) + f(t_n, y(t_n)))(\partial / \partial y)f(t_n, y(t_n)) + O(h^3) + R. \]

Next, we observe that

\[y'(t_n) = f(t_n, y(t_n))\]

by the ODE; differentiating both sides and using the two-variable chain-rule on the right-hand side yields

\[
y''(t_n) = (\partial / \partial t)f(t_n, y(t_n)) + (\partial / \partial y)f(t_n, y(t_n))y'(t_n)
\]

\[= (\partial / \partial t)f(t_n, y(t_n)) + (\partial / \partial y)f(t_n, y(t_n))f(t_n, y(t_n)). \]

Notice that the last line is exactly the term in square brackets in (2). Thus, we can rewrite (2) as

\[y(t_{n+1}) = y(t_n) + hy'(t_n) + (h^2/2)y''(t_n) + O(h^3) + R. \]
Now observe that the first three terms of the right-hand side of the preceding equation are a Taylor expansion in $h$ for $y(t_{n+1})$ about $y(t_n)$, hence the first three terms on the RHS can be replaced by $y(t_{n+1}) - (h^3/6)y'''(t_n) + \cdots$, so the above equation becomes

$$y(t_{n+1}) = y(t_{n+1}) - (h^3/6)y'''(t_n) + O(h^3) + R.$$ 

This shows that $R = O(h^3)$. To get a more precise formula for $R$ (i.e., to obtain the entire leading coefficient), we would need to expand the Taylor series in this derivation to one additional term.