Handed out: Tues., Apr. 3.

This examination lasted 80 minutes and had 80 points total. It was closed book and closed note, but students were permitted to use a $8\frac{1}{2}'' \times 11''$ crib-sheet with notes written on both sides.

1. [5 points] For an $n \times n$ symmetric matrix $A$, what is the definition of “positive definite”?

2. [5 points] Which algorithm should be used to solve the linear least squares problem of minimize $\|Rx - b\|_2$, where $R$ is an $m \times n$ upper triangular matrix with nonzero entries on its main diagonal? How many flops are required (accurate to the leading term)?

3. [10 points] For the function plotted $f(x)$ below, mark the first three iterates of the bisection method (draw three ticks on the x-axis and label them $x_1$, $x_2$, $x_3$) over the interval $[0, 1]$ and circle the root of the function to which the bisection method eventually converges.

4. [10 points] Come up with an example of a $2 \times 2$ matrix $A$ such that $A$ is symmetric, all of its entries are positive, and yet $A$ is not positive definite. Note: for full credit, in addition to providing $A$, also provide a demonstration that $A$ is not positive definite.

5. [10 points] Given a symmetric positive definite $n \times n$ matrix $A$ and a sequence of $n/2$ vectors $b_1, \ldots, b_{n/2}$ each lying in $\mathbb{R}^n$, consider the problem of finding $x_1, \ldots, x_{n/2}$ each in $\mathbb{R}^n$ that solve the linear systems $Ax_1 = b_1, \ldots, Ax_{n/2} = b_{n/2}$. Propose an efficient algorithm for computing $x_1, \ldots, x_{n/2}$, and determine the number of flops (accurate to the leading term) it requires. You may describe your algorithm at a very high level using algorithms described in lecture (such as Cholesky, forward substitution, etc.)
6. **[15 points]** The 1-dimensional case of the “Brouwer fixed point theorem” states that if $f$ is a continuous function from $[0, 1]$ to $[0, 1]$, then there exists a point $x^*$ in $[0, 1]$ such that $f(x^*) = x^*$. Rewrite the problem of finding $x^*$ as a root-finding problem (hint: consider $f(x) - x$), and then argue that the bisection method can find $x^*$, i.e., argue that the necessary conditions for the bisection method are applicable to this problem.

7. **[15 points]** How many square-root operations (accurate to the leading term) are necessary to reduce an $m \times n$ matrix ($m \geq n$) to upper triangular form using Givens rotations? Recall that to compute a single Givens rotation, one square-root operation is required.

8. **[15 points]** What is the solution to the linear least-squares problem $\min \|ax - b\|_2$, where $a, b$ are given $n$-vectors and $x$ is an unknown scalar? Come up with a closed-form solution for $x$. [Hint: consider the normal equations.]