Lecture 23: Graph algorithms

- Topological sort
- Coloring
- Planarity + 4 color theorem

- Quiz
Nodes: tasks  Edges: prerequisites

before starting T1, must complete T2
e.g. tasks are courses.

Topological sort: an ordering of vertices where
if \( i < j \) then there is no edge \( \text{plan}[i] \rightarrow \text{plan}[j] \)
i.e. all prerequisites complete before starting task i.

To start: find a course with no prerequisites

loop invariant:
all outgoing edges in plan only point to vertices already in plan

1. find vertex \( V \) with no outgoing edges
   i.e. with out-degree 0
2. add \( V \) to path,
3. remove \( V \) from \( G \) (and all edges to/from \( V \))
4. topologically sort \( G \)
Graph coloring
- Graph represents a map of countries
- Edge is to color if they share a border
- Goal: assign a color to each country such that adjacent countries have different colors
- Important: find out how many colors are needed?

How to color a graph?
- Pick any vertex, color it
- Pick any other vertex, pick any color different from colors of all neighbors
  (first unused color possible)

Invariants:
- Colored path is a valid coloring

Could have one color per vertex

Question: is there an efficient and optimal algorithm?

Answer: Nobody knows, probably no, if yes, it can be used to solve lots of other problems!

(e.g. decrypt all modern communication)
4-color theorem:
All planar graphs are 4-colorable (i.e. with 4 colors).

Planar: Can be drawn in the plane without edges crossing.

Proof so complicated, needed to be constructed by a computer!