"Simplicity is a great virtue but it requires hard work to achieve it and education to appreciate it. And to make matters worse: complexity sells better."

*- Edsger Dijkstra*

# ASYMPTOTIC COMPLEXITY

Lecture 10

CS2110 – Spring 2019

# Announcements



- Next Mon-Tues: Spring Break
- No recitation next week
- Regrade requests will be processed this weekend
- Prelim is on Tuesday, 12 March. Prelim
  Review for prelim Sunday, 10 March, 1-3PM
  Next Thursday, we will tell you
  - What time you will be assigned to take it
  - What to do if you can't take it then but can take the other one
  - What to do if you can't take it that evening.
  - What to do if authorized for more time or quiet space

# Help in providing code coverage



White-box testing: make sure each part of program is "exercised" in at least one test case. Called code coverage.

Eclipse has a tool for showing you how good your code coverage is! Use it on A3 (and any programs you write)

JavaHyperText entry:

code coverage

We demo it.

# What Makes a Good Algorithm?



Suppose you have two possible algorithms that do the same thing; which is *better*?

What do we mean by *better*?

- Faster?
- Less space?
- Simpler?
- Easier to code?
- Easier to maintain?
- Required for homework?

FIRST, Aim for simplicity, ease of understanding, correctness.

SECOND, Worry about efficiency only when it is needed.

How do we measure speed of an algorithm?

# Basic Step: one “constant time” operation



Constant time operation: its time doesn’t depend on the size or length of anything. Always roughly the same. Time is bounded above by some number

**Basic step:**

- Input/output of a number
- Access value of primitive-type variable, array element, or object field
- assign to variable, array element, or object field ***
- do one arithmetic or logical operation
- method call (not counting arg evaluation and execution of method body)

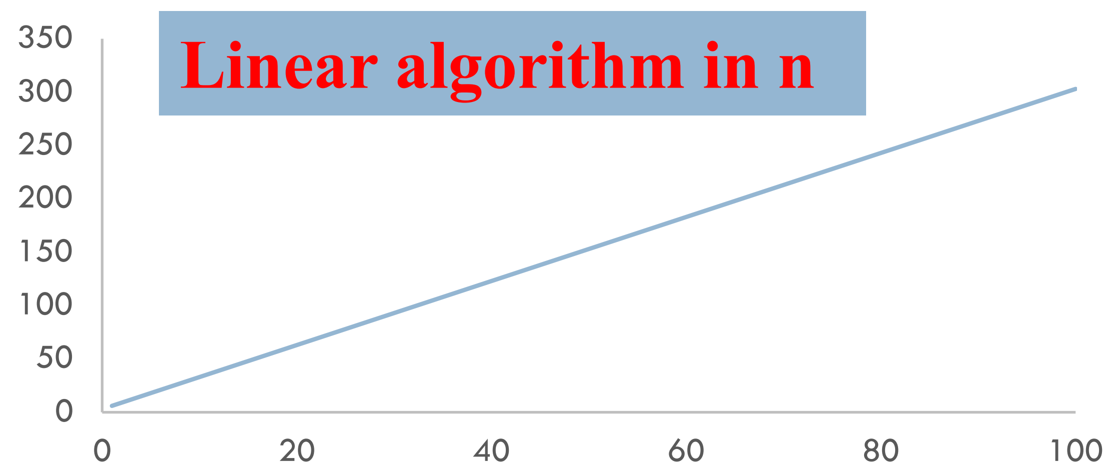
# Counting Steps



```
// Store sum of 1..n in sum
sum= 0;
// inv: sum = sum of 1..(k-1)
for (int k= 1; k <= n; k= k+1){
    sum= sum + k;
}
```

| Statement: | # times done |
| --- | --- |
| sum= 0; | 1 |
| k= 1; | 1 |
| k <= n | n+1 |
| k= k+1; | n |
| sum= sum + k; | n |
| Total steps: | 3n + 3 |

All basic steps take time 1.
There are n loop iterations.
Therefore, takes time proportional to n.



# Not all operations are basic steps



```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k <= n; k= k+1){
    s=  s + 'c';
}
```

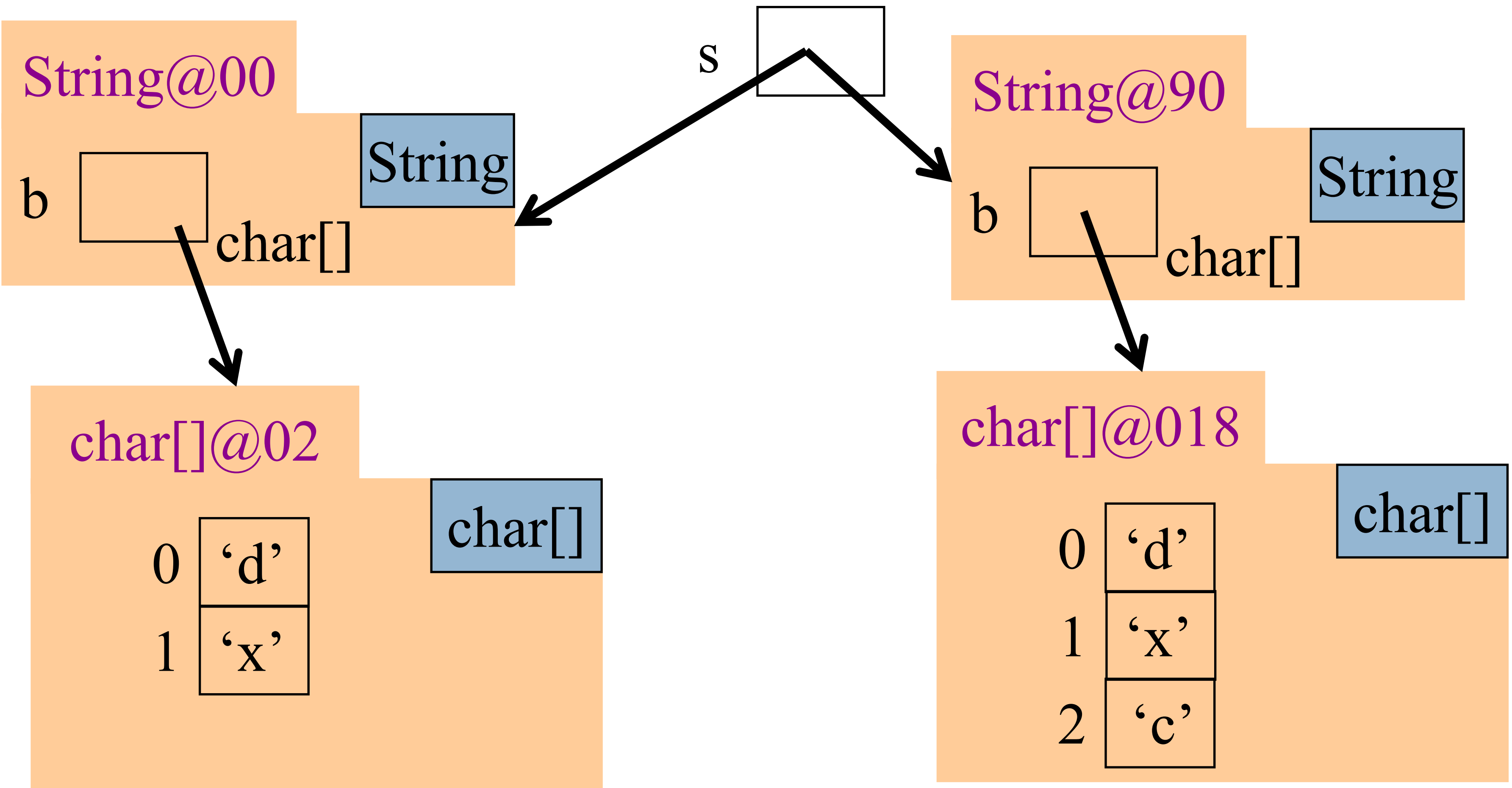
| Statement: | # times done |
|---|---|
| s= ""; | 1 |
| k= 1; | 1 |
| k <= n | n+1 |
| k= k+1; | n |
| s= s + 'c'; | n |
| Total steps: | 3n + 3 |

Catenation is not a basic step.
For each k, catenation creates
and fills k array elements.

# String Catenation



s= s + "c"; is NOT constant time.
It takes time proportional to 1 + length of s



# Basic steps executed in s= s + ‘c’;



s= s + 'c’; // Suppose length of s is k

1. Create new String object, say C basic steps.
2. Copy k chars from object s to the new object: k basic steps
3. Place char ‘c’ into the new object: 1 basic step.
4. Store pointer to new object into s: 1 basic step.

Total of (C+2) + k basic steps.

In the algorithm, s= s + ‘c’; is executed n times:

s= s + ‘c’; with length of s = 0

s= s + ‘c’; with length of s = 1

…

s= s + ‘c’; with length of s = n-1

Total of n*(C+2) + (0 + 1 + 2 + … n-1) basic steps

# Basic steps executed in s= s + ‘c’;



s= s + 'c’; // Suppose length of s is k

In the algorithm, s= s + ‘c’; is executed as follows:

s= s + ‘c’; with length of s = 0

s= s + ‘c’; with length of s = 1

…

s= s + ‘c’; with length of s = n-1

Total of n*(C+2) + (0 + 1 + 2 + … n-1) basic steps

0 + 1 + 2 + … n-1 = n(n-1) / 2. Gauss figured this out in the 1700’s

$= n^2/2 - n/2.$

mathcentral.uregina.ca/qq/database/qq.02.06/jo1.html

# Basic steps executed in s= s + 'c';



s= s + 'c'; // Suppose length of s is k

In the algorithm, s= s + 'c'; is executed as follows:

s= s + 'c'; with length of s = 0

s= s + 'c'; with length of s = 1

…

s= s + 'c'; with length of s = n-1

Total of n*(C+2) + (0 + 1 + 2 + … n-1) basic steps

Total of n*(C+2) + $n^2/2 - n/2$ basic steps

Total of n*(C+2) + $\mathbf{n^2}/2 - n/2$ basic steps. Quadratic in n.

# Not all operations are basic steps

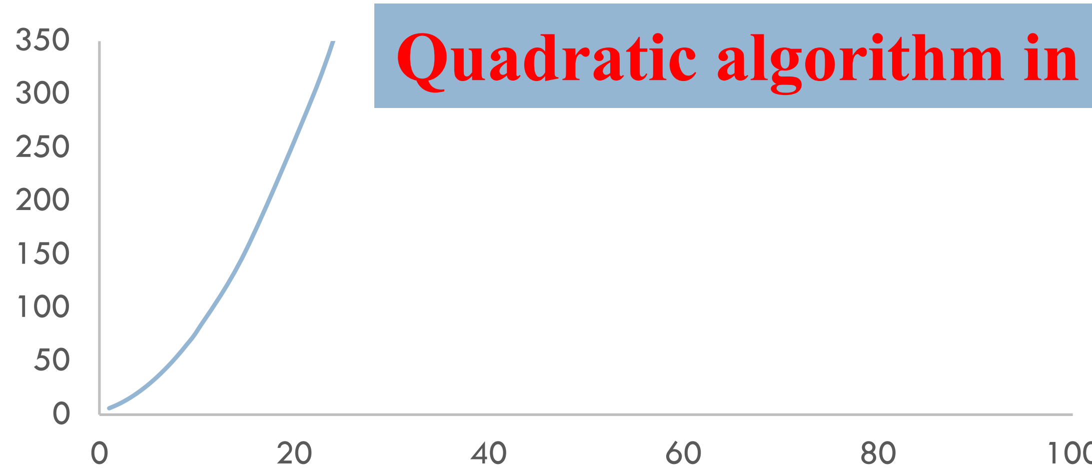


```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k <= n; k= k+1){
    s=  s + 'c';
}
```

| Statement: | # times | # steps |
|---|---|---|
| s= ""; | 1 | 1 |
| k= 1; | 1 | 1 |
| k <= n | n+1 | 1 |
| k= k+1; | n | 1 |
| s= s + 'c'; | see to left | |
| Total steps: | … | |

Total steps:
$2n + 3 +$
$n*(C+2) + \mathbf{n^2}/2 - n/2$

for s= s + 'c';




**Quadratic algorithm in n**

# Linear versus quadractic



```
// Store sum of 1..n in sum
sum= 0;
// inv: sum = sum of 1..(k-1)
for (int k= 1; k <= n; k= k+1)
    sum= sum + n
```

**Linear algorithm**

```
// Store n copies of 'c' in s
s= "";
// inv: s contains k-1 copies of 'c'
for (int k= 1; k = n; k= k+1)
    s=  s + 'c';
```

**Quadratic algorithm**

In comparing the runtimes of these algorithms, the exact number of basic steps is not important. What's important is that

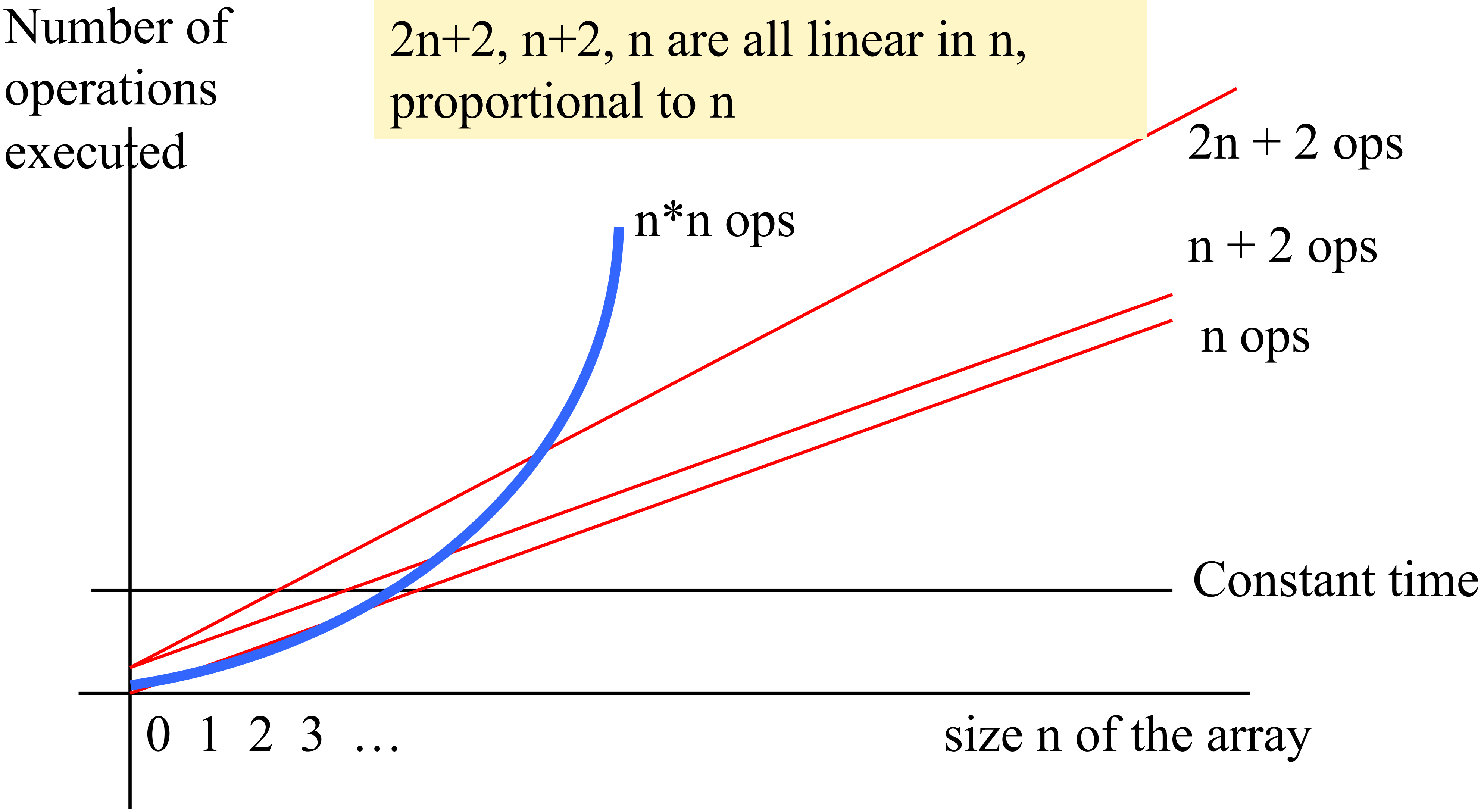
One is linear in n —takes time proportional to n

One is quadratic in n —takes time proportional to $n^2$

# Looking at execution speed

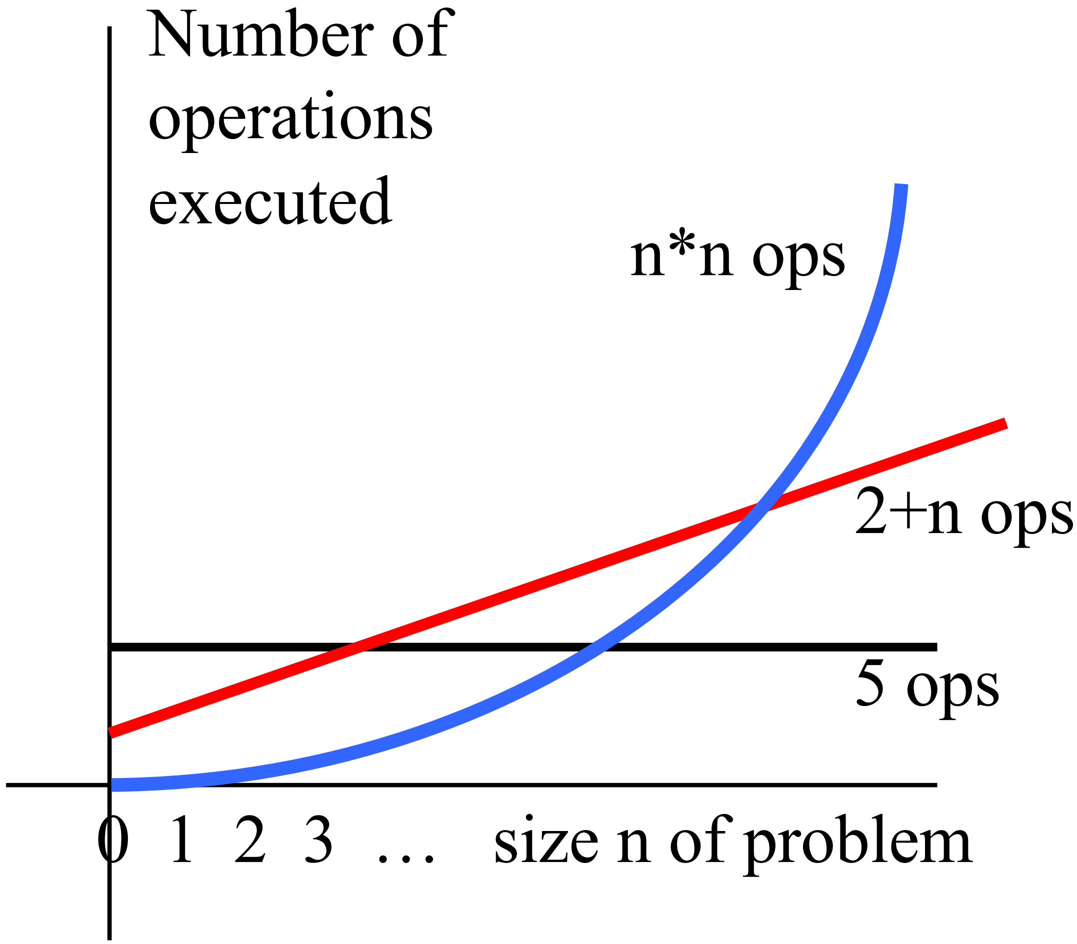


2n+2, n+2, n are all linear in n, proportional to n



# What do we want from a definition of "runtime complexity"?






1. Distinguish among cases for large n, not small n

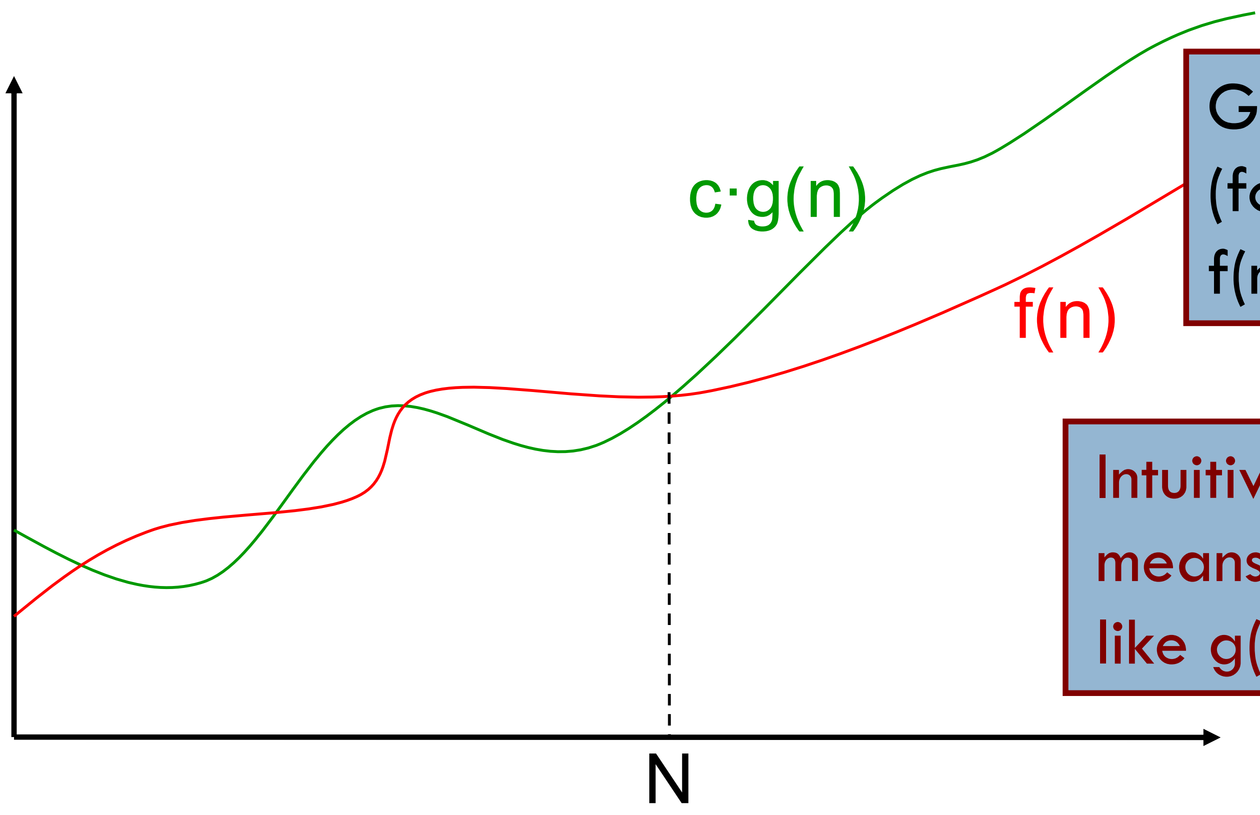
2. Distinguish among important cases, like
- n*n basic operations
- n basic operations
- log n basic operations
- 5 basic operations

3. Don't distinguish among trivially different cases.
- 5 or 50 operations
- n, n+2, or 4n operations

# "Big O" Notation



Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$




Get out far enough (for $n \geq N$) f(n) is at most c·g(n)

Intuitively, f(n) is O(g(n)) means that f(n) grows like g(n) or slower

# Prove that $(2n^2 + n)$ is $O(n^2)$



Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(2n^2 + n)$ is $O(n^2)$

Methodology:

Start with $f(n)$ and slowly transform into $c \cdot g(n)$:

- Use = and <= and < steps
- At appropriate point, can choose N to help calculation
- At appropriate point, can choose c to help calculation

# Prove that $(2n^2 + n)$ is $O(n^2)$



Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

Example: Prove that $(2n^2 + n)$ is $O(n^2)$

$f(n)$

= <definition of f(n)>

$2n^2 + n$

<= <for $n \geq 1$, $n \leq n^2$>

$2n^2 + n^2$

= <arith>

$3*n^2$

= <definition of $g(n) = n^2$>

$3*g(n)$

Transform f(n) into $c \cdot g(n)$:

- Use =, <= , < steps
- Choose N to help calc.
- Choose c to help calc

Choose $N = 1$ and $c = 3$

# Prove that 100 n + log n is O(n)



Formal definition: f(n) is O(g(n)) if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

f(n)

= <put in what f(n) is>

100 n + log n

<= <We know log n ≤ n for n ≥ 1>

100 n + n

= <arith>

101 n

= <g(n) = n>

101 g(n)

Choose
N = 1 and c = 101

# O(…) Examples



Let $f(n) = 3n^2 + 6n - 7$

- $f(n)$ is $O(n^2)$
- $f(n)$ is $O(n^3)$
- $f(n)$ is $O(n^4)$
- …

$p(n) = 4 n \log n + 34 n - 89$

- $p(n)$ is $O(n \log n)$
- $p(n)$ is $O(n^2)$

$h(n) = 20 \cdot 2^n + 40n$

$h(n)$ is $O(2^n)$

$a(n) = 34$

- $a(n)$ is $O(1)$

Only the *leading* term (the term that grows most rapidly) matters

If it's $O(n^2)$, it's also $O(n^3)$ etc! However, we always use the smallest one

# Do NOT say or write $f(n) = O(g(n))$



Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $N \geq 0$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

$f(n) = O(g(n))$ is simply WRONG. Mathematically, it is a disaster. You see it sometimes, even in textbooks. Don't read such things.

Here's an example to show what happens when we use = this way.

We know that $n+2$ is $O(n)$ and $n+3$ is $O(n)$. Suppose we use =

$$n+2 = O(n)$$
$$n+3 = O(n)$$

But then, by transitivity of equality, we have $n+2 = n+3$.

We have proved something that is false. Not good.

# Problem-size examples



- Suppose a computer can execute 1000 operations per second; how large a problem can we solve?

| operations | 1 second | 1 minute | 1 hour |
|---|---|---|---|
| n | 1000 | 60,000 | 3,600,000 |
| n log n | 140 | 4893 | 200,000 |
| $n^2$ | 31 | 244 | 1897 |
| $3n^2$ | 18 | 144 | 1096 |
| $n^3$ | 10 | 39 | 153 |
| $2^n$ | 9 | 15 | 21 |

# Commonly Seen Time Bounds



| | | |
|---|---|---|
| $O(1)$ | constant | excellent |
| $O(\log n)$ | logarithmic | excellent |
| $O(n)$ | linear | good |
| $O(n \log n)$ | n log n | pretty good |
| $O(n^2)$ | quadratic | maybe OK |
| $O(n^3)$ | cubic | maybe OK |
| $O(2^n)$ | exponential | too slow |

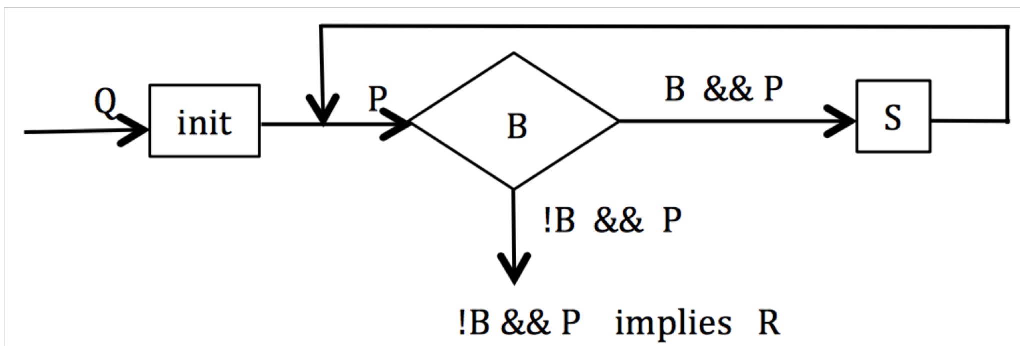
# Search for v in b[0..]



Q: v is in array b

Store in i the index of the first occurrence of v in b:

R: v is not in b[0..i-1] and b[i] = v.




Methodology:

1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

**Practice doing this!**

# Search for v in b[0..]



Q: v is in array b

Store in i the index of the first occurrence of v in b:

R: v is not in b[0..i-1] and b[i] = v.

pre: b

| 0 | b.length |
|---|---|
| v in here | |

post: b

| 0 | i | | b.length |
|---|---|---|---|
| $\neq v$ | $v$ | ? | |

inv: b

| 0 | i | b.length |
|---|---|---|
| $\neq v$ | v in here | |

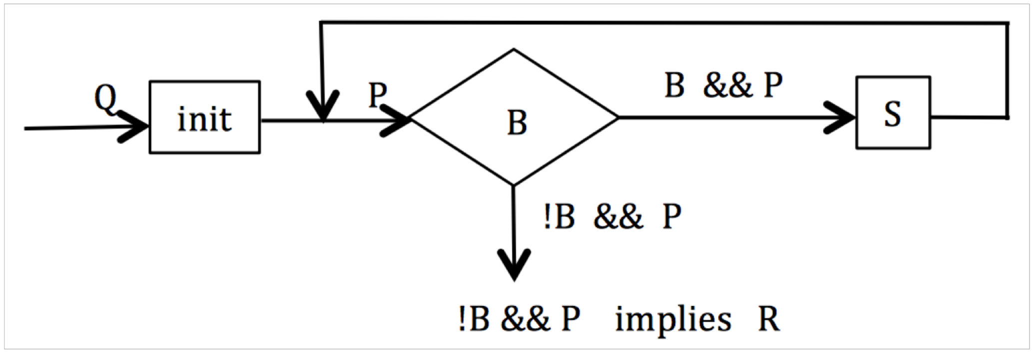
Methodology:

1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

**Practice doing this!**

# The Four Loopy Questions






- Does it start right?

  Is {Q} init {P} true?

- Does it continue right?

  Is {P && B} S {P} true?

- Does it end right?

  Is P && !B => R true?

- Will it get to the end?

  Does it make progress toward termination?

# Search for v in b[0..]



Q: v is in array b

Store in i the index of the first occurrence of v in b:

R: v is not in b[0..i-1] and b[i] = v.

pre: b

| 0 | | b.length |
|---|---|---|
| v in here | | |

post: b

| 0 | i | b.length |
|---|---|---|
| $\neq v$ | $v$ ? | |

inv: b

| 0 | i | b.length |
|---|---|---|
| $\neq v$ | v in here | |

```
i= 0;
while (b[i] != v) {
    i= i+1;
}
```

Each iteration takes constant time.

Worst case: b.length iterations

**Linear algorithm: O(b.length)**

# Binary search for v in sorted b[0..]

```
// b is sorted. Store in i  a value to truthify R:
//        b[0..i] <= v < b[i+1..]
```

pre: b

| 0 | b.length |
|---|---|
| sorted | |

post: b

| 0 | i | b.length |
|---|---|---|
| $\leq v$ | $> v$ | |

inv: b

| 0 | i | k | b.length |
|---|---|---|---|
| $\leq v$ | ? | $> v$ | |

b is sorted. We know that. To avoid clutter, don't write in it invariant

Methodology:

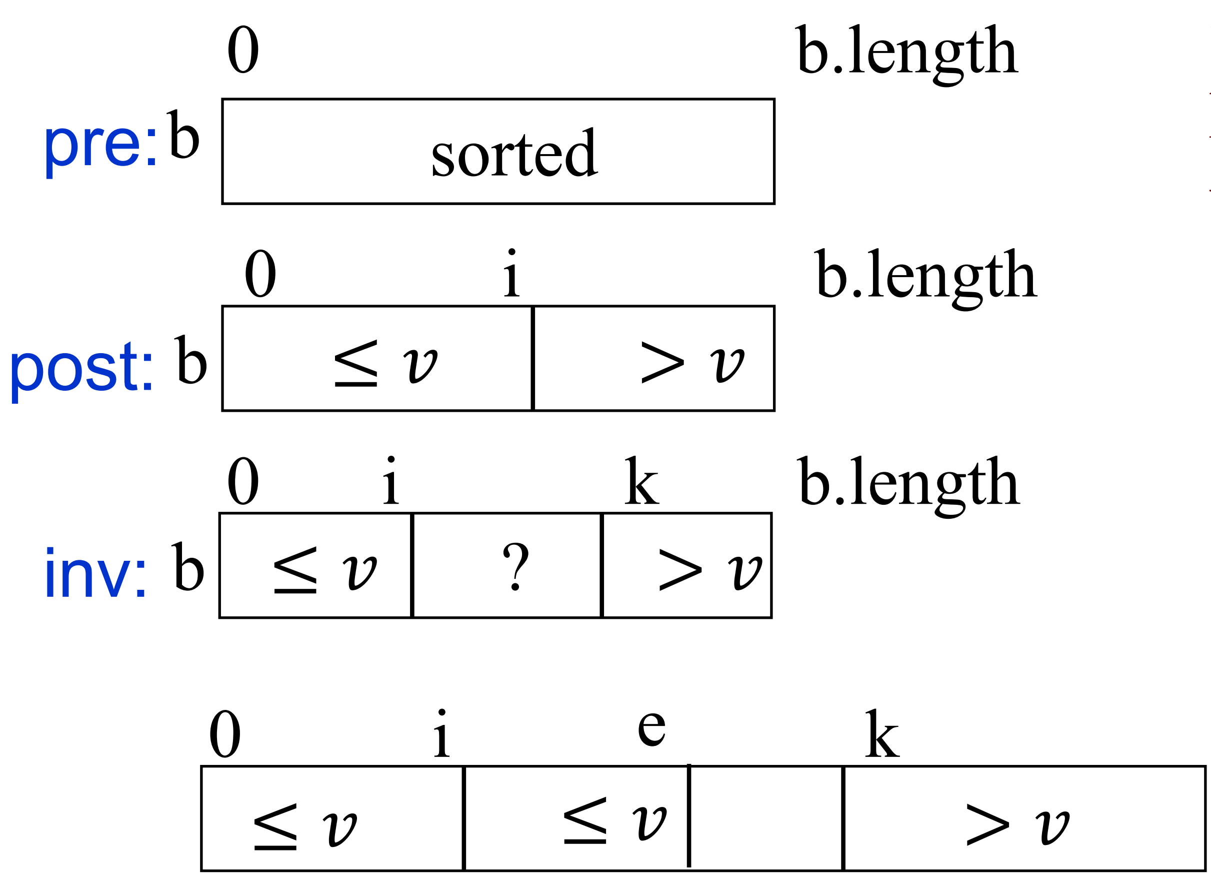
1. Define pre and post conditions.
2. Draw the invariant as a combination of pre and post.
3. Develop loop using 4 loopy questions.

**Practice doing this!**

# Binary search for v in sorted b[0..]



// b is sorted. Store in i a value to truthify R:
//        b[0..i] <= v < b[i+1..]




```
i= -1;
k= b.length;
while ( i+1< k) {
   int e=(i+k)/2;
   // -1 ≤ i < e < k ≤ b.length
   if (b[e] <= v)  i= e;
   else k= e;
}
```

# Binary search for v in sorted b[0..]



```
// b is sorted. Store in i  a value to truthify R:
//         b[0..i] <= v < b[i+1..]
```

| | 0 | | | b.length |
|---|---|---|---|---|
| pre: b | sorted | | | |

| | 0 | i | | b.length |
|---|---|---|---|---|
| post: b | $\leq v$ | | $> v$ | |

| | 0 | i | k | b.length |
|---|---|---|---|---|
| inv: b | $\leq v$ | ? | $> v$ | |

```
i= -1;
k= b.length;
while  ( i+1< k) {
   int e=(i+k)/2;
   // -1 ≤ e < k ≤ b.length
   if (b[e] <= v)  i= e;
   else k= e;
}
```

Each iteration takes constant time.

**Logarithmic: O(log(b.length))**

Worst case:
log(b.length) iterations

# Binary search for v in sorted b[0..]



```
// b is sorted. Store in i  a value to truthify R:
//        b[0..i] <= v < b[i+1..]
```

This algorithm is better than binary searches that stop when v is found.

1. Gives good info when v not in b.
2. Works when b is empty.
3. Finds first occurrence of v, not arbitrary one.
4. Correctness, including making progress, easily seen using invariant

```
i= -1;
k= b.length;
while  ( i+1< k) {
   int e=(i+k)/2;
   // -1 ≤ e < k ≤ b.length
   if (b[e] <= v)  i= e;
   else k= e;
}
```

Each iteration takes constant time.

**Logarithmic: O(log(b.length))**

Worst case:
log(b.length) iterations

# Dutch National Flag Algorithm



# Dutch National Flag Algorithm

**Dutch national flag**. Swap b[0..n-1] to put the reds first, then the whites, then the blues. That is, given precondition Q, swap values of b[0.n-1] to truthify postcondition R:

| | 0 | | | n |
|---|---|---|---|---|
| Q: b | ? | | | |

| | 0 | | n |
|---|---|---|---|
| R: b | reds | whites | blues |

| | 0 | | | n |
|---|---|---|---|---|
| P1: b | reds | whites | blues | ? |

| | 0 | | | n |
|---|---|---|---|---|
| P2: b | reds | whites | ? | blues |

Suppose we use invariant P1.

What does the repetend do?

2 swaps to get a red in place

# Dutch National Flag Algorithm

**Dutch national flag**. Swap b[0..n-1] to put the reds first, then the whites, then the blues. That is, given precondition Q, swap values of b[0.n-1] to truthify postcondition R:

Q: b (0 … n)

| ? |
|---|

R: b (0 … n)

| reds | whites | blues |
|---|---|---|

P1: b (0 … n)

| reds | whites | blues | ? |
|---|---|---|---|

P2: b (0 … n)

| reds | whites | ? | blues |
|---|---|---|---|

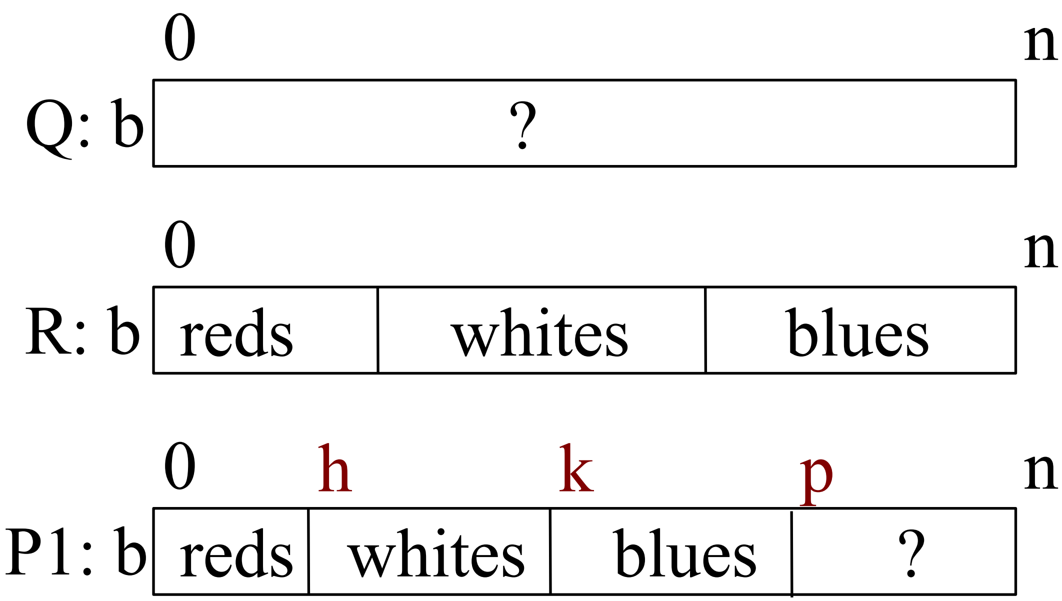
Suppose we use invariant P2.

What does the repetend do?

At most one swap per iteration

Compare algorithms without writing code!

# Dutch National Flag Algorithm: invariant P1




```
h= 0; k= h; p= k;
while ( p != n ) {
    if (b[p] blue)   p= p+1;
    else if (b[p] white) {
            swap b[p], b[k];
            p= p+1; k= k+1;
    }
    else { // b[p] red
            swap b[p], b[h];
            swap b[p], b[k];
            p= p+1; h=h+1; k= k+1;
    }
}
```

# Dutch National Flag Algorithm: invariant P2

| | 0 | n |
|---|---|---|
| Q: b | ? | |

| | 0 | | | n |
|---|---|---|---|---|
| R: b | reds | whites | blues | |

| | 0 | h | k | p | n |
|---|---|---|---|---|---|
| P2: b | reds | whites | ? | blues | |

```
h= 0; k= h; p= n;
while ( k != p ) {
    if (b[k] white)  k=  k+1;
    else if (b[k] blue) {
        p= p-1;
        swap b[k], b[p];
    }
    else { // b[k] is red
        swap b[k], b[h];
        h= h+1; k= k+1;
    }
}
```

# Asymptotically, which algorithm is faster?



## Invariant 1

0 h k p n

| reds | whites | blues | ? |
|---|---|---|---|

```
h= 0; k= h; p= k;
while ( p != n ) {
    if (b[p] blue)        p=  p+1;
    else if (b[p] white) {
        swap b[p], b[k];
        p= p+1; k= k+1;
    }
    else { // b[p] red
        swap b[p], b[h];
        swap b[p], b[k];
        p= p+1; h=h+1; k= k+1;
    }
}
```

## Invariant 2

0 h k p n

| reds | whites | ? | blues |
|---|---|---|---|

```
h= 0; k= h; p= n;
while ( k != p ) {

    if (b[k] white)        k=  k+1;
    else if (b[k] blue) {
        p= p-1;
        swap b[k], b[p];
    }
    else { // b[k] is red
        swap b[k], b[h];
        h= h+1; k= k+1;
    }
}
```

# Asymptotically, which algorithm is faster?



**Invariant 1**

| 0 | h | k | p | n |
|---|---|---|---|---|
| reds | whites | blues | ? | |

might use 2 swaps per iteration

```
if (b[p] blue)          p= p+1;
else if (b[p] white) {
        swap b[p], b[k];
        p= p+1; k= k+1;
```

```
        swap b[p], b[h];
        swap b[p], b[k];
        p= p+1; h=h+1; k= k+1;
    }
}
```

**Invariant 2**

| 0 | h | k | p | n |
|---|---|---|---|---|
| reds | whites | ? | blues | |

uses at most 1 swap per iteration

```
if (b[k] white)          k= k+1;
else if (b[k] blue) {
        p= p-1;
```

```
        swap b[k], b[h];
        h= h+1; k= k+1;
    }
}
```

These two algorithms have the same asymptotic running time (both are O(n))