The invariant of Dijkstra’s shortest-path algorithm

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The shortest-path algorithm is a breadth-first search algorithm. It has a loop, and each iteration of the loop will identify the shortest distance to one more node. We now present the invariant of the loop and prove a simple theorem about it.

The set of all nodes is partitioned into three sets: a settled set $S$ (red), a frontier set $F$ (blue), and a far-off set (black).

The invariant consists of three points:

1. For a settled node $s$, at least one shortest path from $v$ to $s$ contains only settled nodes, and $d[s]$ is the distance of that shortest path from $v$ to $s$.

Think of the settled set as places that we know all about because we have visited often and settled there.

A node in the frontier has been visited at least once but not enough to know for certain that its shortest-path distance has been fully calculated. Think of the frontier as the moon and close planets that we know something about, but not everything. We have not settled there yet. Here’s the second part of the invariant:

2. For a node $f$ in the frontier, at least one path from $v$ to $f$ contains only settled nodes, except for the last one, $f$ (as shown below), and $d[f]$ is the minimum distance of all such paths.

That is, over paths that start with settled node $v$, perhaps contain more red nodes, and finally have one edge to $f$. There is a degenerate case: when $f$ and $v$ are the same node and it is in the frontier set. This degenerate case will make it easy to initialize the invariant, including putting $v$ into the frontier set.

Here’s the third part of the invariant:

3. All edges leaving the settled set end in the frontier set.

That’s all there is to the invariant! Three simple and easy-to-remember points.

Stop the video at this point and convince yourself that if $v$ is in either the settled or the frontier set, the invariant implies that $d[v] = 0$.

A theorem based on the invariant. We now prove an important theorem.

Theorem. For a node $f$ in the frontier with minimum $d$ value (over nodes in the frontier), $d[f]$ is indeed the shortest-path distance from $v$ to $f$.

For example, suppose the frontier contains three nodes $f_0$, $f_1$, and $f_2$, with shortest-distances $d[f_0] = 5$, $d[f_1] = 4$, and $d[f_2] = 6$. Then $f_1$ is the node in the frontier with minimum $d$ value.

We prove this theorem by showing that any other path from $v$ to $f$ does not have a smaller distance. We consider two cases:

Case 1. $v$ and $f$ are the same, and that node is in the frontier set. First, the settled set is empty, for, by invariant P1, if a node $s$ is in the settled set than $v$ must be there too, but it isn’t; it is in the frontier. Second, since $v$ is in the frontier set, by P2, the frontier set contains only one node, $v$. Third, we know that $d[v]$ is 0, and 0 is the distance of the shortest path from $v$ to $v$.

Case 2. $v$ is in the settled set. Suppose that the shown path from $v$ to $f$ is the one with distance $d[f]$. By part 2 of the invariant, $d[f]$ is the shortest-path distance over paths that contain only settled (red) nodes except for $f$. Consider any other $v$-$f$ path. It starts at $v$, goes through settled nodes, visits $f$ (so that path had
distance $\geq d[f]$) or visits another frontier node $g$ (say) and then winds its way to $f$. Since $d[g] \geq d[f]$, and since edge weights are positive, the distance of this path is $\geq d[f]$.

Q.E.D. (Quit-End-Done)