Spanning Trees

Lecture 22
CS2110 – Spring 2017
Prelim 2, assignments

• Prelim 2 is Tuesday. See the course webpage for details.
• Scope: up to but not including today’s lecture. See the review guide for details.
  – Deadline for submitting conflicts has passed.

• A6 was due last night. Late penalty 3 points per day, up to 3 days. No exceptions – the solution is used in A7!

• A7 due next Thursday 4/27. It’s short; 30-40 lines including comments. Do it before the prelim and it doubles as studying Dijkstra!
A Note on Dijkstra

1. For s, d[s] is length of shortest v → s path.

2. For f, d[f] is length of shortest v → f path of form

   ![Diagram showing nodes and paths]

3. Edges leaving S go to F.

Theorem: For a node f in F with min d value, d[f] is its shortest path length

```
S = { }; F = { v }; d[v] = 0;
while F ≠ {} {
    f = node in F with min d value;
    Remove f from F, add it to S;
    for each neighbor w of f {
        if (w not in S or F) {
            d[w] = d[f] + wgt(f, w);
            add w to F;
        } else
            if (d[f] + wgt(f, w) < d[w]) {
                d[w] = d[f] + wgt(f, w);
            }
    }
}
```
Undirected trees

An undirected graph is a \textit{tree} if there is exactly one simple path between any pair of vertices.

What’s the root? It doesn’t matter! Any vertex can be root.
Facts about trees

• $\#E = \#V - 1$
• connected
• no cycles

Any two of these properties imply the third and thus imply that the graph is a tree
Spanning trees

A *spanning tree* of a connected undirected graph \((V, E)\) is a subgraph \((V, E')\) that is a tree.

- Same set of vertices \(V\)
- \(E' \subseteq E\)
- \((V, E')\) is a tree

Three equivalent definitions:

- Same set of vertices \(V\)
- Maximal set of edges that contains no cycle

- Same set of vertices \(V\)
- Minimal set of edges that connect all vertices
Spanning trees: examples

http://mathworld.wolfram.com/SpanningTree.html
Finding a spanning tree: **Subtractive method**

- Start with the whole graph – it is connected
- While there is a cycle:
  - Pick an edge of a cycle and throw it out
  - the graph is still connected (why?)

Maximal set of edges that contains no cycle

nondeterministic algorithm

One step of the algorithm
Aside: How can you find a cycle in an undirected graph?
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/** Visit all nodes REACHABLE* from u. Pre: u is unvisited. */

public static void dfs(int u) {
    Stack s = (u);
    while (s is not empty) {
        u = s.pop();
        if (u has not been visited) {
            visit u;
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
}
Aside: How can you find a cycle in an undirected graph?

/** True if the nodes reachable from u have a cycle. */

public static boolean hasCycle(int u) {
    Stack s = new Stack();
    while (s is not empty) {
        u = s.pop();
        if (u has been visited) {
            return true;
        } else {
            visit u;
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
    return false;
}
Finding a spanning tree: **Subtractive method**

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- While there is a cycle:
  - Pick an edge of a cycle and throw it out
  - the graph is still connected (why?)

Maximal set of edges that contains no cycle

nondeterministic algorithm

One step of the algorithm
Finding a spanning tree: Additive method

- Start with no edges
- While the graph is not connected:
  Choose an edge that connects 2 connected components and add it
  – the graph still has no cycle (why?)

Tree edges will be red.
Dashed lines show original edges.
Left tree consists of 5 connected components, each a node
Aside: How do you find connected components?
Aside: How do you find connected components?

/** Visit all nodes REACHABLE* from u. Pre: u is unvisited. */

public static void dfs(int u) {
    Stack s = (u);
    while (s is not empty) {
        u = s.pop();
        if (u has not been visited) {
            visit u;
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
}
Aside: How do you find connected components?

/** Return the set of nodes in u’s connected component. */
public static Set<int> getComponent(int u) {
    Stack s = (u);
    Set C = ();
    while (s is not empty) {
        u = s.pop();
        if (u has not been visited) {
            visit u;
            C.add(u);
            for each edge (u, v) leaving u:
                s.push(v);
        }
    }
    return C;
}
Finding a spanning tree: Additive method

- Start with no edges

- While the graph is not connected:
  Choose an edge that connects 2 connected components and add it
  – the graph still has no cycle (why?)

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Dashed lines show original edges.
Left tree consists of 5 connected components, each a node.
Spanning trees: examples

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Minimum spanning trees

• Suppose edges are weighted (> 0)
• We want a spanning tree of *minimum cost* (sum of edge weights)

• Some graphs have exactly one minimum spanning tree. Others have several trees with the same minimum cost, each of which is a minimum spanning tree

• Useful in network routing & other applications. For example, to stream a video
Greedy algorithm

A greedy algorithm follows the heuristic of making a locally optimal choice at each stage, with the hope of finding a global optimum.

Example. Make change using the fewest number of coins. Make change for n cents, $n < 100$ (i.e. $< $1)
Greedy: At each step, choose the largest possible coin

If $n \geq 50$ choose a half dollar and reduce $n$ by 50;
If $n \geq 25$ choose a quarter and reduce $n$ by 25;
As long as $n \geq 10$, choose a dime and reduce $n$ by 10;
If $n \geq 5$, choose a nickel and reduce $n$ by 5;
Choose $n$ pennies.
Greedy algorithm — doesn’t always work!

A greedy algorithm follows the heuristic of making a locally optimal choice at each stage, with the hope of finding a global optimum. Doesn’t always work

Example. Make change using the fewest number of coins. Coins have these values: 7, 5, 1
Greedy: At each step, choose the largest possible coin

Consider making change for 10.
The greedy choice would choose: 7, 1, 1, 1.
But 5, 5 is only 2 coins.
Greediness doesn’t work here

You’re standing at point x, and your goal is to climb the highest mountain.

Two possible steps: down the hill or up the hill. The greedy step is to walk up hill. But that is a local optimum choice, not a global one. Greediness fails in this case.
Finding a minimal spanning tree

Suppose edges have > 0 weights

**Minimal spanning tree**: sum of weights is a minimum

We show two greedy algorithms for finding a minimal spanning tree.

They are versions of the basic additive method we have already seen: at each step add an edge that does not create a cycle.

Kruskal: add an edge with minimum weight. Can have a forest of trees.

Prim (JPD): add an edge with minimum weight but so that the added edges (and the nodes at their ends) form *one* tree
**MST using Kruskal’s algorithm**

At each step, add an edge (that does not form a cycle) with minimum weight

- Edge with weight 2
- Edge with weight 3

One of the 4’s

Red edges need not form tree (until end)
Kruskal

Start with the all the nodes and no edges, so there is a forest of trees, each of which is a single node (a leaf).

At each step, add an edge (that does not form a cycle) with minimum weight.

We do not look more closely at how best to implement Kruskal’s algorithm — which data structures can be used to get a really efficient algorithm.

Leave that for later courses, or you can look them up online yourself.

We now investigate Prim’s algorithm.
MST using “Prim’s algorithm” (should be called “JPD algorithm”)
**Prim’s algorithm**

At each step, add an edge (that does not form a cycle) with minimum weight, but keep added edge connected to the start (red) node.

- **Edge with weight 3**: The edge added with weight 3.
- **Edge with weight 5**: The edge added with weight 5.

**One of the 4’s**

- The 2

Minimal set of edges that connect all vertices
**Difference between Prim and Kruskal**

Prim requires that the constructed red tree always be connected.
Kruskal doesn’t

But: Both algorithms find a minimal spanning tree

<table>
<thead>
<tr>
<th>Vertex</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge</td>
<td></td>
<td>4</td>
<td></td>
<td></td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, Prim chooses (0, 1)
Kruskal chooses (3, 4)

Here, Prim chooses (0, 2)
Kruskal chooses (3, 4)

Minimal set of edges that connect all vertices
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Difference between Prim and Kruskal

Prim requires that the constructed red tree always be connected.
Kruskal doesn’t

But: Both algorithms find a minimal spanning tree

If the edge weights are all different, the Prim and Kruskal algorithms construct the same tree.
Prim’s (JPD) spanning tree algorithm

Given: graph \((V, E)\)  (sets of vertices and edges)
Output: tree \((V_1, E_1)\), where

\[ V_1 = V \]

\[ E_1 \text{ is a subset of } E \]

\((V_1, E_1)\) is a minimal spanning tree – sum of edge weights is minimal
Prim’s (JPD) spanning tree algorithm

\[ V_1 = \{ \text{an arbitrary node of } V \} ; \quad E_1 = \{ \} ; \]

//inv: (V_1, E_1) is a tree, \( V_1 \leq V, E_1 \leq E \)

\[
\text{while} \ (V_1.\text{size()}) < (V.\text{size()}) \ \{ \\
\quad \text{Pick an edge } (u,v) \text{ with:} \\
\quad \quad \text{min weight, } u \text{ in } V_1, \\
\quad \quad v \text{ not in } V_1; \\
\quad \text{Add } v \text{ to } V_1; \\
\quad \text{Add edge } (u,v) \text{ to } E_1 \\
\}
\]

Consider having a set \( S \) of edges with the property:
If \((u,v)\) an edge with \( u \) in \( V_1 \) and \( v \) not in \( V_1 \), then \((u,v)\) is in \( S \)

- \( V_1 \): 2 red nodes
- \( E_1 \): 1 red edge
- \( S \): 2 edges leaving red nodes
Prim’s (JPD) spanning tree algorithm

\[ V_1 = \{ \text{an arbitrary node of } V \}; \quad E_1 = \{ \}; \]

//inv: \((V_1, E_1)\) is a tree, \(V_1 \leq V, E_1 \leq E\)

\[
\text{while } (V_1.\text{size()} < V.\text{size()} ) \{ \\
\quad \text{Pick an edge } (u,v) \text{ with:} \\
\quad \quad \text{min weight, } u \text{ in } V_1, \\
\quad \quad v \text{ not in } V_1; \\
\quad \text{Add } v \text{ to } V_1; \\
\quad \text{Add edge } (u,v) \text{ to } E_1 \\
\}
\]

Consider having a set \(S\) of edges with the property:
If \((u,v)\) an edge with \(u\) in \(V_1\) and \(v\) not in \(V_1\), then \((u,v)\) is in \(S\)
Prim’s (JPD) spanning tree algorithm

\[ V_1 = \{ \text{an arbitrary node of } V \}; \quad E_1 = \{ \}; \]

//inv: \( (V_1, E_1) \) is a tree, \( V_1 \leq V, E_1 \leq E \)

while \( \text{V1.size()} < \text{V.size()} \) {
    Pick an edge \((u, v)\) with:
    \[
    \begin{align*}
    & \text{min weight, } u \text{ in } V_1, \\
    & v \text{ not in } V_1;
    \end{align*}
    \]
    Add \( v \) to \( V_1 \);
    Add edge \((u, v)\) to \( E_1 \)
}

Consider having a set \( S \) of edges with the property:
If \((u, v)\) an edge with \( u \) in \( V_1 \) and \( v \) not in \( V_1 \), then \((u,v)\) is in \( S \)

V1: 4 red nodes
E1: 3 red edges
S: 3 edges leaving red nodes

Note: the edge with weight 6 is not in \( S \) – this avoids cycles.
Prim’s (JPD) spanning tree algorithm

V1= \{an arbitrary node of V\}; E1= \{\};
//inv: (V1, E1) is a tree, V1 ≤ V, E1 ≤ E
S= set of edges leaving the single node in V1;
while (V1.size() < V.size()) {
    Pick an edge (u,v) with:
    --min weight, u in V1;
    --v not in V1;
    Add v to V1;
    Add edge (u, v) to E1;
    Remove from S an edge (u, v) with min weight
    if v is not in V1: add v to V1; add (u,v) to E1;
    add edges leaving v to S
}

Consider having a set S of edges with the property:
If (u, v) an edge with u in V1 and v not in V1, then (u,v) is in S
Prim’s (JPD) spanning tree algorithm

\[ V_1 = \{ \text{start node} \}; \quad E_1 = \{ \}; \]
\[ S = \text{set of edges leaving the single node in } V_1; \]

// inv: \((V_1, E_1)\) is a tree, \(V_1 \leq V, E_1 \leq E,\)
// All edges \((u, v)\) in \(S\) have \(u\) in \(V_1,\)
// if edge \((u, v)\) has \(u\) in \(V_1\) and \(v\) not in \(V_1, (u, v)\) is in \(S\)

while \((V_1.\text{size()} < V.\text{size()}\)) {
    Remove from \(S\) an edge \((u, v)\) with min weight;
    if \((v\) not in \(V_1)\) {
        add \(v\) to \(V_1\); add \((u,v)\) to \(E_1\);
        add edges leaving \(v\) to \(S\)
    }
}

Question: How should we implement set \(S\)?
Prim’s (JPD) spanning tree algorithm

\( V_1 = \{ \text{start node} \}; \quad E_1 = \{ \}; \)

\( S = \text{set of edges leaving the single node in } V_1; \)

//inv: \((V_1, E_1)\) is a tree, \(V_1 \leq V, E_1 \leq E,\)

// All edges \((u, v)\) in \(S\) have \(u\) in \(V_1,\)

// if edge \((u, v)\) has \(u\) in \(V_1\) and \(v\) not in \(V_1,\) \((u, v)\) is in \(S\)

\textbf{while} \((V_1.\text{size()} < V.\text{size()})) \{ \)

\hspace{1em} \text{Remove from } S \text{ a min-weight edge } (u, v); \quad \#V \log \#E

\hspace{1em} \text{if } (v \text{ not in } V_1) \{ \)

\hspace{2em} \text{add } v \text{ to } V_1; \text{ add } (u,v) \text{ to } E_1; \quad \#E \log \#E

\hspace{2em} \text{add edges leaving } v \text{ to } S

\}

\}

Implement \(S\) as a heap.

Use adjacency lists for edges

Thought: Could we use for \(S\) a set of nodes instead of edges? Yes. We don’t go into that here
Application of minimum spanning tree

Maze generation using Prim’s algorithm

The generation of a maze using Prim's algorithm on a randomly weighted grid graph that is 30x20 in size.

https://en.wikipedia.org/wiki/Maze_generation_algorithm
#Randomized_Kruskal.'s_algorithm
In this undirected graph, all edge weights are 1. **Which of the following visit the nodes in the same order as Prim(1)?**

- Always break ties by choosing the lower-numbered node first.
- In tree traversals, use node 1 as the tree’s root.

- Dijkstra(1)
- BFS(1)
- DFS(1)
- Preorder tree traversal
- Postorder tree traversal
- Level order tree traversal
Greedy algorithms

Suppose the weights are all 1. Then Dijkstra’s shortest-path algorithm does a breath-first search!

Dijkstra’s and Prim’s algorithms look similar. The steps taken are similar, but at each step
• Dijkstra’s chooses an edge whose end node has a minimum path length from start node
• Prim’s chooses an edge with minimum length
Breadth-first search, Shortest-path, Prim

**Greedy algorithm**: An algorithm that uses the heuristic of making the locally optimal choice at each stage with the hope of finding the global optimum.

Dijkstra’s shortest-path algorithm makes a locally optimal choice: choosing the node in the Frontier with minimum L value and moving it to the Settled set. And, it is proven that it is not just a hope but a fact that it leads to the global optimum.

Similarly, Prim’s and Kruskal’s locally optimum choices of adding a minimum-weight edge have been proven to yield the global optimum: a minimum spanning tree.

**BUT**: Greediness does not always work!
while (a vertex is unmarked) {
    v = best unmarked vertex
    mark v;
    for (each w adj to v)
        update D[w];
}

• Breadth-first-search (bfs)
  – best: next in queue
  – update: D[w] = D[v] + 1

• Dijkstra’s algorithm
  – best: next in priority queue
  – update: D[w] = min(D[w], D[v] + c(v, w))

• Prim’s algorithm
  – best: next in priority queue
  – update: D[w] = min(D[w], c(v, w))

\[ c(v, w) \] is the \[ v \rightarrow w \] edge weight
Traveling salesman problem

Given a list of cities and the distances between each pair, what is the shortest route that visits each city exactly once and returns to the origin city?

– The true TSP is very hard (called NP complete)… for this we want the *perfect* answer in all cases.

– Most TSP algorithms start with a spanning tree, then “evolve” it into a TSP solution. Wikipedia has a lot of information about packages you can download…

But really, how hard can it be?

How many paths can there be that visit all of 50 cities?

\[12,413,915,592,536,072,670,862,289,047,373,375,038,521,486,354,677,760,000,000,000\]
Graph Algorithms

• Search
  – Depth-first search
  – Breadth-first search
• Shortest paths
  – Dijkstra's algorithm
• Minimum spanning trees
  – Prim's algorithm
  – Kruskal's algorithm