Fibonacci
(Leonardo Pisano)
1170-1240?
Statue in Pisa Italy

FIBONACCI NUMBERS
GOLDEN RATIO,
RECURRENCES

Lecture 23
CS2110 – Fall 2016

Fibonacci
Leonardo Pisano
1170-1240?
Statue in Pisa Italy

Fibonacci function
*** Return fib(n). Precondition: n ≥ 0.***
public static int f(int n) {
    if ( n <= 1) return n;
    return f(n-1) + f(n-2);
}

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
We’ll see that this is a lousy way to compute f(n)

Golden ratio Φ = (1 + √5)/2 = 1.61803398⋯
Find the golden ratio when we divide a line into two parts a and b such that
(a + b) / a = a / b
= Φ
See webpage:
http://www.mathsisfun.com/numbers/golden-ratio.html

Prelim tonight
You know already whether you are taking it at 5:30 or 7:30.
5:30 Exam:
A thru Te...
Take in Uris Hall G01
Th... thru Z.
Take it in Ives 305

7:30 Exam:
A... thru De...
Take it in Ives 305
Dh... thru Z.
Take it in Uris Hall G01

Statue in Pisa Italy

Fibonacci function (year 1202)
*** Return fib(n). Precondition: n ≥ 0.***
public static int f(int n) {
    if ( n <= 1) return n;
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Golden ratio $\Phi = \frac{1 + \sqrt{5}}{2} = 1.61803398\ldots$

Find the golden ratio when we divide a line into two parts $a$ and $b$ such that

$$\frac{a + b}{a} = \frac{a}{b} = \Phi$$

For successive Fibonacci numbers $a$, $b$, $a/b$ is close to $\Phi$ but not quite $\Phi$. 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Find $\text{fib}(n)$ from $\text{fib}(n-1)$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

Since $\text{fib}(n) / \text{fib}(n-1)$ is close to the golden ratio,

You can see that $(\text{golden ratio}) * \text{fib}(n-1)$ is close to $\text{fib}(n)$

We can actually use this formula to calculate $\text{fib}(n)$ from $\text{fib}(n-1)$

Golden ratio and Fibonacci numbers: inextricably linked

The Parthenon

Fibonacci function (year 1202)

Downloaded from wikipedia

Fibonacci tiling

Fibonacci spiral

0, 1, 1, 2, 3, 5, 8, 13, 21, 34 …

The golden ratio

How to draw a golden rectangle

Fibonacci and bees

Male bee has only a mother
Female bee has mother and father

The number of ancestors at any level is a Fibonacci number

MB: male bee,  FB: female bee
Fibonacci in Pascal's Triangle

\[ p(i,j) \] is the number of ways \( i \) elements can be chosen from a set of size \( j \)

Suppose you are a plant

You want to grow your leaves so that they all get a good amount of sunlight. You decide to grow them at successive angles of 180 degrees.

Pretty stupid plant! The two bottom leaves get VERY little sunlight!

Suppose you are a plant

You want to grow your leaves so that they all get a good amount of sunlight. 90 degrees, maybe?

Where does the fifth leaf go?

Fibonacci in nature

The artichoke uses the Fibonacci pattern to spiral the sprouts of its flowers.

\[
\text{golden ratio} = \frac{360}{222.5} = 222.492
\]

The artichoke sprouts its leafs at a constant amount of rotation: 222.5 degrees (in other words the distance between one leaf and the next is 222.5 degrees).

topones.weebly.com/1/post/2012/10/the-artichoke-and-fibonacci.html

Blooms: strobe-animated sculptures

www.instructables.com/id/Blooming-Zoetrope-Sculptures/

Uses of Fibonacci sequence in CS

Fibonacci search
Fibonacci heap data structure
Fibonacci cubes: graphs used for interconnecting parallel and distributed systems
/** Return fib(n). Precondition: n ≥ 0. */

public static int f(int n) {
    if ( n <= 1 ) return n;
    return f(n-1) + f(n-2);
}

Recursion for fib: \( f(n) = f(n-1) + f(n-2) \)

- **T(0) = a**
- **T(1) = a**
- **T(n) = T(n-1) + T(n-2)**

We can prove that T(n) is \( O(2^n) \)

It’s a “proof by induction”. Proof by induction is not covered in this course. But we can give you an idea about why T(n) is \( O(2^n) \)

Recursion for fib: \( f(n) = f(n-1) + f(n-2) \)

- **T(0) = a**
- **T(1) = a**
- **T(n) = T(n-1) + T(n-2)**

We can go on forever like this
Recursion for fib: \( f(n) = f(n-1) + f(n-2) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( T(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a )</td>
</tr>
<tr>
<td>1</td>
<td>( a )</td>
</tr>
<tr>
<td>( n )</td>
<td>( T(n) = T(n-1) + T(n-2) )</td>
</tr>
</tbody>
</table>

- \( T(0) = a \leq a \times 2^0 \)
- \( T(1) = a \leq a \times 2^1 \)
- \( T(2) \leq a \times 2^2 \)
- \( T(3) \leq a \times 2^3 \)
- \( T(4) \leq a \times 2^4 \)

The golden ratio

- \( a > 0 \) and \( b > a > 0 \) are in the golden ratio if
- \( (a + b) / b = b / a \)
- \( \phi^2 = \phi + 1 \) so \( \phi = (1 + \sqrt{5}) / 2 = 1.618 \ldots \)
- \( \phi \) ratio of sum of sides to longer side
- \( b \) ratio of longer side to shorter side

Linear algorithm to calculate fib(n)

```
/** Return fib(n), for n >= 0. */
public static int f(int n) {
    if (n <= 1) return 1;
    int p= 0;   int c= 1;  int i= 2;
    // invariant: p = fib(i-2) and c = fib(i-1)
    while (i < n) {
        int fibi= c + p;   p= c;  c= fibi;
        i= i+1;
    }
    return c + p;
}
```

Caching

As values of \( f(n) \) are calculated, save them in an ArrayList. Call it a cache.

When asked to calculate \( f(n) \) see if it is in the cache. If yes, just return the cached value. If no, calculate \( f(n) \), add it to the cache, and return it.

Must be done in such a way that if \( f(n) \) is about to be cached, \( f(0), f(1), \ldots, f(n-1) \) are already cached.

Can prove that Fibonacci recurrence is \( O(\phi^n) \)

We won’t prove it.
Requires proof by induction
Relies on identity \( \phi^2 = \phi + 1 \)

Logarithmic algorithm!

\[
\begin{pmatrix} f_0 & f_1 \\ f_{n+2} & f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix}
\]

\[
\begin{pmatrix} f_{n+k} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} f_n & f_{n+1} & \cdots & f_{n+k-1} \end{pmatrix}
\]
Logarithmic algorithm!

\[ f_0 = 0 \]
\[ f_1 = 1 \]
\[ f_{n+2} = f_{n+1} + f_n \]

Define \( \phi = \frac{1 + \sqrt{5}}{2} \) and \( \phi' = \frac{1 - \sqrt{5}}{2} \)

The golden ratio again.

Prove by induction on \( n \) that

\[ f_n = \frac{\phi^n - \phi'^n}{\sqrt{5}} \]

You know a logarithmic algorithm for exponentiation—recursive and iterative versions.

Gries and Levin
Computing a Fibonacci number in log time.
IPL 2 (October 1980), 68-69.

Another log algorithm!

Define \( \phi = \frac{1 + \sqrt{5}}{2} \) and \( \phi' = \frac{1 - \sqrt{5}}{2} \)

You know a logarithmic algorithm for exponentiation—recursive and iterative versions.

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