Recitation 9

Analysis of Algorithms and Inductive Proofs

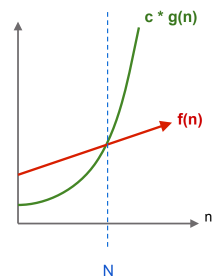
Review: Big O definition

f(n) is O(g(n))

iff

There exists c > 0 and N > 0 such that:

f(n) ≤ c * g(n) for n ≥ N



Example: n+6 is O(n)

n + 6 ---this is f(n)
<= <if 6 <= n, write as>
n + n
= <arith>
2*n
<choose c = 2>
= c*n ---this is c * g(n)

So choose c = 2 and N = 6

f(n) is O(g(n)): There exist c > 0, N > 0 such that:
f(n) ≤ c * g(n) for n ≥ N

Review: Big O

Is used to classify algorithms by how they respond to changes in input size n.

Important vocabulary:

- Constant time: O(1)
- Logarithmic time: O(log n)
- Linear time: O(n)
- Quadratic time: O(n^2)
- Exponential time: O(2^n)

Review: Big O

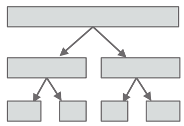
1. log(n) + 20	is	O(log(n))	(logarithmic)
2. n + log(n)	is	O(n)	(linear)
3. n/2 and 3*n	are	O(n)	
4. n * log(n) + n	is	O(n * log(n))	
5. n^2 + 2*n + 6	is	O(n^2)	(quadratic)
6. n^3 + n^2	is	O(n^3)	(cubic)
7. 2^n + n5	is	O(2^n)	(exponential)

Merge Sort

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);
      mS(b, e+1, k);
      merge(b, h, e, k);
}

mS is mergeSort for readability
```



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mS is mergeSort for readability
```

- We will *count* the number of comparisons mS makes
- Use `T(n)` for the number of array element comparisons that mS makes on an array segment of size *n*

Runtime of merge sort

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      merge(b, h, e, k);
}
```

T(0) = 0
T(1) = 0

Use `T(n)` for the number of array element comparisons that mergeSort makes on an array of size *n*

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);        T(e+1-h) comparisons = T(n/2)
      mS(b, e+1, k);      T(k-e)   comparisons = T(n/2)
      merge(b, h, e, k);           How long does merge take?
}
```


Runtime of merge

```
pseudocode for merge
/** Pre: b[h..e] and b[e+1..k] are already sorted */
merge(Comparable[] b, int h, int e, int k)
    Copy both segments
    While both copies are non-empty
        Compare the first element of each segment
        Set the next element of b to the smaller value
        Remove the smaller element from its segment

                One comparison, one add, one remove

k-h loops must empty one segment
```

Runtime is O(k-h)

Runtime of merge sort

```
/** Sort b[h..k]. */
public static void mS(Comparable[] b, int h, int k) {
      if (h >= k) return;
      int e= (h+k)/2;
      mS(b, h, e);        T(e+1-h) comparisons = T(n/2)
      mS(b, e+1, k);      T(k-e)   comparisons = T(n/2)
      merge(b, h, e, k);  O(k-h)   comparisons = O(n)
}
```

Recursive Case:
T(n) = 2T(n/2) + O(n)

Runtime

We determined that

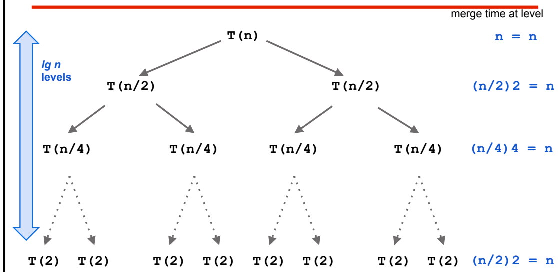
$T(1) = 0$

$T(n) = 2T(n/2) + n$ for $n > 1$

We will prove that

$T(n) = n \log_2 n$ (or $n \lg n$ for short)

Recursion tree



lg n levels * n comparisons is O(n log n)

Proof by induction

To prove $T(n) = n \lg n$,
we can assume true for smaller values of n (like recursion)

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2(n/2)\lg(n/2) + n \\
&= n(\lg n - \lg 2) + n \\
&= n(\lg n - 1) + n \\
&= n \lg n - n + n \\
&= n \lg n
\end{aligned}
$$

Property of logarithms

$\log_2 2 = 1$

Heap Sort

Heap Sort

Very simple idea:
1. Turn the array into a max-heap
2. Pull each element out

```
/** Sort b */
public static void heapSort(Comparable[] b) {
    heapify(b);
    for (int i= b.length-1; i >= 0; i--) {
        b[i]= poll(b, i);
    }
}
```


Heap Sort

```
/** Sort b */
public static void heapSort(Comparable[] b) {
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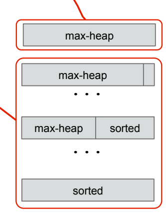

Why does it have to be a max-heap?

Heap Sort runtime

```
/** Sort b */
public static void heapSort(Comparable[] b) {
    heapify(b);                                    O(n lg n)
    for (int i= b.length-1; i >= 0; i--) {
        b[i]= poll(b, i);                          loops n times
    }
}       O(lg n)
```

Total runtime:

O(n lg n) + n*O(lg n) = O(n lg n)