RECURSION

Lecture 8
CS2110 – Fall 2016

Three things

- Note: We've covered almost every-thing in JavaSummary.pptx!
- Recursion? 7.1-7.39 slide 1-7
- Base case 7.1-7.10 slide 13
- How Java stack frames work 7.8-7.10 slide 28-32

Supplemental material
Pinned Piazza note @281

- Java concepts …
- Study habits
- Wrapper classes
- Recursion exercises
- Recitation solutions
- …

ABOUT PRELIM 1

Visit the exams page of the course website:
http://www.cs.cornell.edu/courses/CS2110/2016fa/exams.html
Download the study guide.
Read about time assignments and how to handle conflicts

Why flip recitations?

Usual way. 50-minute lecture, then study on your own. One hour? Total of, say, 2 hours.

Disadvantages:

- Hard to listen attentively for 50 minutes. Many people tune out, look at internet, videos, whatever
- Much time wasted here and there
- You don't always know just how to study. No problem sets, and if there are, no easy way to check answers.
- Study may consist of reading, not doing. Doesn't help.

Why flip recitations?

Flipped way. Watch short, usually 3-5 minute, videos on a topic. Then come to recitation and participate in solving problems.

Disadvantage: If you don't study the videos carefully, you are wasting your time.

Advantages

- Break up watching videos into shorter time periods.
- Watch parts of one several times.
- In recitation, you get to DO something, not just read, and you get to discuss with a partner and neighbors, ask TA questions, etc.

One student's reaction, in an email

… I really enjoyed today's activity and found it extremely effective in gaining a strong understanding of the material. The act of discussing problems with fellow classmates made me aware of what topics I was not as strong in and gave me the opportunity to address those areas immediately. What I enjoyed most about it, however, was working collaboratively with my peers.

I wanted to give you my feedback because I can see these interactive lessons becoming very effective if implemented again in the future.

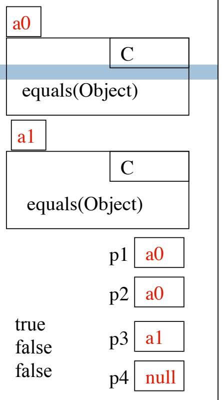
== versus equals

Use p1 == p2 or p1 != p2 to determine whether p1 and p2 point to the same object (or are both null).

NEVER use p1.equals(p2) for this purpose, because it doesn't always tell whether they point to the same object! It depends on how equals is defined.

p4.equals(p1)
Null pointer exception!

p2 == p1	true
p3 == p1	false
p4 == p1	false



Sum the digits in a non-negative integer

7

```
/** = sum of digits in n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

   // { n has at least two digits }
   // return first digit + sum of rest
   return n%10  +  sum(n/10) ;
}
```

sum calls itself!

E.g. sum(7) = 7

E.g. sum(8703) = 3 + sum(870);

Two issues with recursion

8

```
/** return sum of digits in n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

   // { n has at least two digits }
   // return first digit + sum of rest
   return n%10  +  sum(n/10);
}
```

sum calls itself!

1. Why does it work? How does execution work?

2. How do we understand a given recursive method or how do we write/develop a recursive method?

Stacks and Queues

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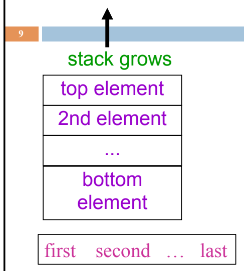

Stack: list with (at least) two basic ops:
- * Push an element onto its top
- * Pop (remove) top element

Last-In-First-Out (LIFO)

Like a stack of trays in a cafeteria

Queue: list with (at least) two basic ops:
- * Append an element
- * Remove first element

First-In-First-Out (FIFO)

Americans wait in a line. The Brits wait in a queue !

Stack Frame

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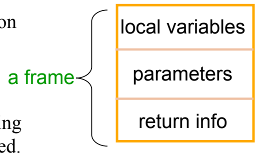
A "frame" contains information about a method call:

At runtime Java maintains a stack that contains frames for all method calls that are being executed but have not completed.



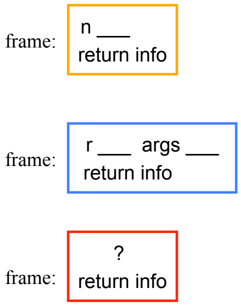
Method call: push a frame for call on stack. Assign argument values to parameters. Execute method body. Use the frame for the call to reference local variables and parameters.

End of method call: pop its frame from the stack; if it is a function leave the return value on top of stack.

Frames for methods sum main method in the system

11

```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
        String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```



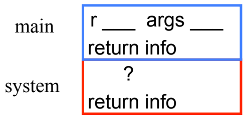
Frame for method in the system that calls method main

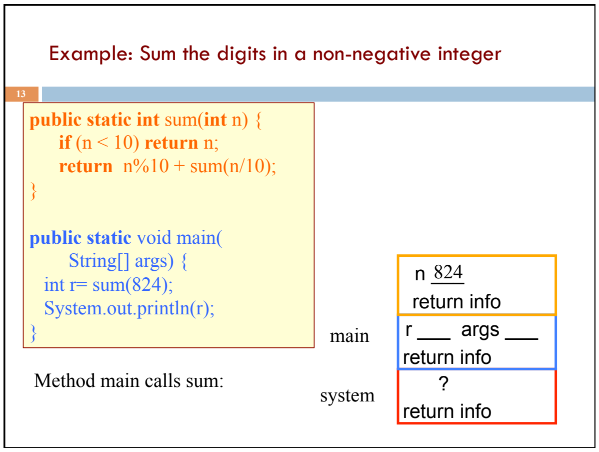
Example: Sum the digits in a non-negative integer

12

```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10  +  sum(n/10);
}

public static void main(
        String[] args) {
  int r= sum(824);
  System.out.println(r);
}
```

Frame for method in the system that calls method main: main is then called





Example: Sum the digits in a non-negative integer

13

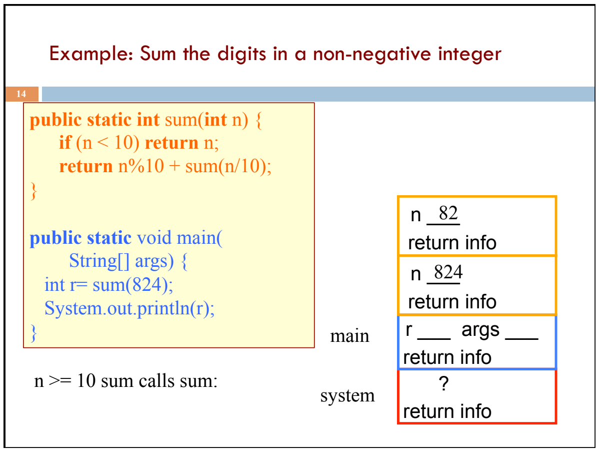
```
public static int sum(int n) {
    if (n < 10) return n;
    return  n%10 + sum(n/10);
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

n 824
return info

main
r ___ args ___
return info

system
?
return info

Method main calls sum:



Example: Sum the digits in a non-negative integer

14

```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

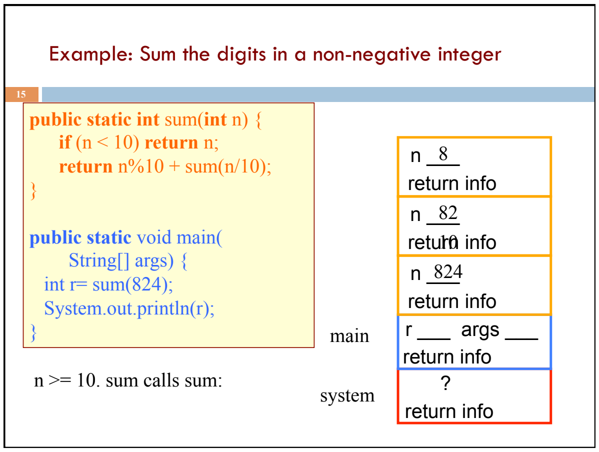
n 82
return info

n 824
return info

main
r ___ args ___
return info

system
?
return info

n >= 10 sum calls sum:



Example: Sum the digits in a non-negative integer

15

```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

n 8
return info

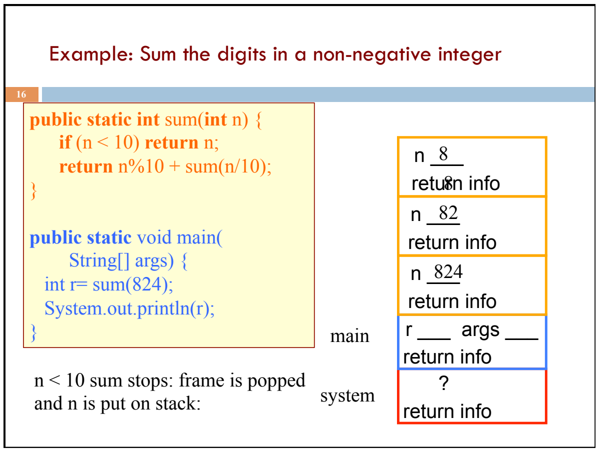
n 82
return info
10

n 824
return info

main
r ___ args ___
return info

system
?
return info

n >= 10. sum calls sum:



Example: Sum the digits in a non-negative integer

16

```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

n 8
return info
8

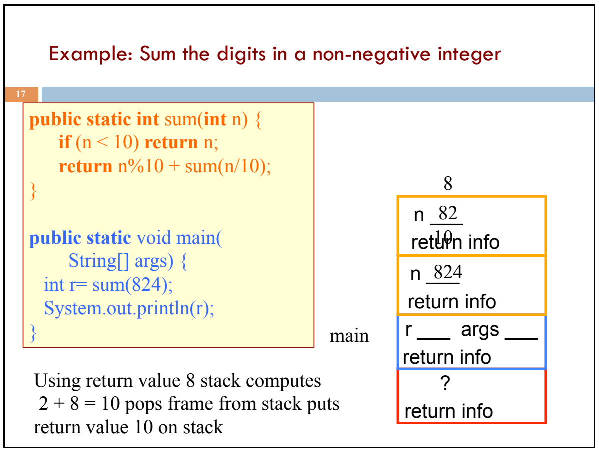
n 82
return info

n 824
return info

main
r ___ args ___
return info

system
?
return info

n < 10 sum stops: frame is popped
and n is put on stack:



Example: Sum the digits in a non-negative integer

17

```
public static int sum(int n) {
    if (n < 10) return n;
    return n%10 + sum(n/10);
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

8

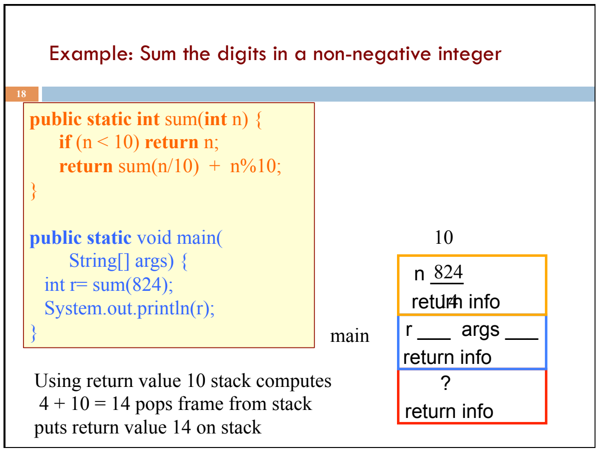
n 82
return info
10

n 824
return info

main
r ___ args ___
return info

?
return info

Using return value 8 stack computes
2 + 8 = 10 pops frame from stack puts
return value 10 on stack



Example: Sum the digits in a non-negative integer

18

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
    int r= sum(824);
    System.out.println(r);
}
```

10

n 824
return info
14

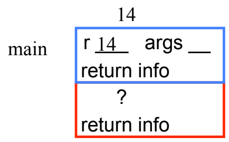
main
r ___ args ___
return info

?
return info

Using return value 10 stack computes
4 + 10 = 14 pops frame from stack
puts return value 14 on stack

Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10)  +  n%10;
}

public static void main(
        String[] args) {
   int r= sum(824);
   System.out.println(r);
}
```



Using return value 14 main stores
14 in r and removes 14 from stack

Memorize method call execution!

A frame for a call contains parameters, local variables, and other information needed to properly execute a method call.

To execute a method call:

1. push a frame for the call on the stack,
2. assign argument values to parameters,
3. execute method body,
4. pop frame for call from stack, and (for a function) push returned value on stack

When executing method body look in frame for call for parameters and local variables.

Questions about local variables

```
public static void m(…) {
   …
   while (…) {
      int d= 5;
      …
   }
}
```

```
public static void m(…) {
   int d;
   …
   while (…) {
      d= 5;
      …
   }
}
```

In a call m(…)
when is local variable d created and when is it destroyed?
Which version of procedure m do you like better? Why?

Recursion is used extensively in math

Math definition of n factorial

$0! = 1$

$n! = n * (n-1)!$ for $n > 0$

E.g. $3! = 3*2*1 = 6$

Math definition of b^c for $c >= 0$

$b^0 = 1$

$b^c = b * b^{c-1}$ for $c > 0$

Easy to make math definition into a Java function!

```
public static int fact(int n) {
  if (n == 0) return 1;

  return n * fact(n-1);
}
```

Lots of things defined recursively:
expression grammars trees ….
We will see such things later

Two views of recursive methods

- How are calls on recursive methods executed?
  We saw that. Use this only to gain understanding / assurance that recursion works
- How do we understand a recursive method — know that it satisfies its specification? How do we write a recursive method?
  This requires a totally different approach. Thinking about how the method gets executed will confuse you completely! We now introduce this approach.

How to understand what a call does

Make a copy of the method spec, replacing the parameters of the method by the arguments

spec says that the value of a call equals the sum of the digits of n

sum(654)

sum of digits of **n**

sum of digits of **654**

```
/** =  sum of the digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return n%10  + sum(n/10);
}
```

Understanding a recursive method

25

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s): That is, Cases where the parameter is small enough that the result can be computed simply and without recursive calls.

```
/** =  sum of digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return n%10  + sum(n/10);
}
```

If n < 10 then n consists of a single digit. Looking at the spec we see that that digit is the required sum.

Understanding a recursive method

26

```
/** =  sum of digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return n%10  + sum(n/10);
}
```

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the method spec and verify that the correct result is then obtained.

return n%10 + sum(n/10);

return n%10 + (sum of digits of n/10); // e.g. n = 843

Understanding a recursive method

27

```
/** =  sum of digits of n.
  * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;

    // n has at least two digits
    return n%10  + sum(n/10);
}
```

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does acc. to the spec and verify correctness.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

n/10 < n

Understanding a recursive method

28

Step 1. Have a precise spec!

Important! Can't do step 3 without precise spec.

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the spec and verify correctness.

Once you get the hang of it this is what makes recursion easy! This way of thinking is based on math induction which we don't cover in this course.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the parameters of the method

Writing a recursive method

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Step 1. Have a precise spec!

Step 2. Write the base case(s): Cases in which no recursive calls are needed. Generally for "small" values of the parameters.

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind. Done suitably, point 4 (about termination) is automatically satisfied.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the parameters of the method

Examples of writing recursive functions

30

For the rest of the class we demo writing recursive functions using the approach outlined below. The java file we develop will be placed on the course webpage some time after the lecture.

Step 1. Have a precise spec!

Step 2. Write the base case(s).

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind.

The Fibonacci Function

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Mathematical definition:

fib(0) = 0 ← two base cases!
fib(1) = 1
fib(n) = fib(n − 1) + fib(n − 2) n ≥ 2

Fibonacci sequence: 0 1 1 2 3 5 8 13 …

```
/** = fibonacci(n). Pre: n >= 0 */
static int fib(int n) {
   if (n <= 1) return n;
   // { 1 < n }
   return fib(n-1) + fib(n-2);
}
```

Fibonacci (Leonardo Pisano) 1170-1240?

Statue in Pisa Italy
Giovanni Paganucci
1863

Check palindrome-hood

32

A String palindrome is a String that reads the same backward and forward.

A String with at least two characters is a palindrome if

- □ (0) its first and last characters are equal and
- □ (1) chars between first & last form a palindrome:

have to be the same

e.g. AMANAPLANACANALPANAMA

have to be a palindrome

A recursive definition!

Example: Is a string a palindrome?

33

```
/** = "s is a palindrome" */
public static boolean isPal(String s) {
    if (s.length() <= 1)
        return true;

    // { s has at least 2 chars }
    int n= s.length()-1;
    return s.charAt(0) == s.charAt(n)  &&  isPal(s.substring(1,n));
}
```

Substring from s[1] to s[n-1]

isPal("racecar") returns true
isPal("pumpkin") returns false

- □ A man a plan a caret a ban a myriad a sum a lac a liar a hoop a pint a catalpa a gas an oil a bird a yell a vat a caw a pax a wag a tax a nay a ram a cap a yam a gay a tsar a wall a car a luger a ward a bin a woman a vassal a wolf a tuna a nit a pall a fret a watt a bay a daub a tan a cab a datum a gall a hat a fag a zap a say a jaw a lay a wet a gallop a tug a trot a trap a tram a torr a caper a top a tonk a toll a ball a fair a sax a minim a tenor a bass a passer a capital a rut an amen a ted a cabal a tang a sun an ass a maw a sag a jam a dam a sub a salt an axon a sail an ad a wadi a radian a room a rood a rip a tad a pariah a revel a reel a reed a pool a plug a pin a peek a parabola a dog a pat a cud a nu a fan a pal a rum a nod an eta a lag an eel a batik a mug a mot a nap a maxim a mood a leek a grub a gob a gel a drab a citadel a total a cedar a tap a gag a rat a manor a bar a gal a cola a pap a yaw a tab a raj a gab a nag a pagan a bag a jar a bat a way a papa a local a gar a baron a mat a rag a gap a tar a decal a tot a led a tic a bard a leg a bog a burg a keel a doom a mix a map an atom a gum a kit a baleen a gala a ten a don a mural a pan a faun a ducat a pagoda a lob a rap a keep a nip a gulp a loop a deer a leer a lever a hair a pad a tapir a door a moor an aid a raid a wad an alias an ox an atlas a bus a madam a jag a saw a mass an anus a gnat a lab a cadet an em a natural a tip a caress a pass a baronet a minimax a sari a fall a ballot a knot a pot a rep a carrot a mart a part a tort a gut a poll a gateway a law a jay a sap a zag a fat a hall a gamut a dab a can a tabu a day a batt a waterfall a patina a nut a flow a lass a van a mow a nib a draw a regular a call a war a stay a gam a yap a cam a ray an ax a tag a wax a paw a cat a valley a drib a lion a saga a plat a catnip a pooh a rail a calamus a dairyman a bater a canal Panama

Example: Count the e's in a string

35

```
/** =  number of times c occurs in s */
public static int countEm(char c, String s) {
   if (s.length() == 0) return 0;

   // { s has at least 1 character }
   if (s.charAt(0) != c)
       return countEm(c, s.substring(1));

   // { first character of s is c}
   return 1 + countEm (c, s.substring(1));
}
```

substring s[1..]
i.e. s[1] …
s(s.length()-1)

- □ countEm('e', "it is easy to see that this has many e's") = 4
- □ countEm('e', "Mississippi") = 0