CORRECTNESS ISSUES
AND LOOP INVARIANTS
The next several lectures

Study algorithms for searching and sorting arrays. Investigate their complexity – how much time and space they take. “Formalize” the notions of average-case and worst-case complexity.

We want you to know these algorithms:
- *Not* by memorizing code but by being able to develop the algorithms from their specifications and, when necessary, a small idea.

We give you some guidelines and instructions on how to develop an algorithm from its specification. Deal mainly with developing loops.
Many (most) of you could use instruction on developing algorithms, keeping things simple

String[] dummy = s.split("\"\"\"\"\") // turns s into string array
int len = s.length()-1; // length of string s
String a = "\"\"\""; // will be reverse of s

for (int b = len; b > -1; b--){
    a = a.dummy[b];
}

if (s.equals(a)) return true;
else return false;

Why calculate the reverse of s?
Some principles and strategies for development

• Don’t introduce a variable without a good reason.
• Put local variables as close to their first use as possible.
• Structure expressions to make them readable.
• Make the structure of the program reflect the structure of the data.
• Never have lots of syntax errors.
• Intersperse coding and testing: code a little, test a little.
• Write the class invariant while putting in field declarations.
• Write a method spec before writing the method body.
• Use assert statements to check method preconditions –as along as it doesn’t complicate program too much and doesn’t change the time-complexity of the method.
Show *development* of *isPalindrome*

```java
/** Return true iff s is a palindrome */
public static boolean isPalindrome(String s)

Our instructions said to visit each char of s only once!
```
isPalindrome: Set ispal to “s is a palindrome” (forget about returns for now. Store value in ispal.

Think of checking equality of outer chars, then chars inside them, then chars inside them, etc.

Key idea: Generalize this to a picture that is true before/after each iteration
isPalindrome: Set ispal to “s is a palindrome”
(forget about returns for now. Store value in ispal.

Generalize to a picture that is true before/after each iteration

These sections are each others’ reverse
isPalindrome: Set ispal to “s is a palindrome”

```java
int h = 0;
int k = s.length() - 1;
// s[0..h-1] is the reverse of s[k+1..]
while (h < k && s.charAt(h) == s.charAt(k)) {
    h = h + 1;
    k = k - 1;
}
ispal = h >= k;
```

Initialization to make picture true
Stop when result is known
Continue when it’s not
Make progress toward termination
AND keep picture true

These sections are each others’ reverse
/** Return true iff s is a palindrome */

```java
global static boolean isPal(String s) {
    int h= 0; int k= s.length() – 1;
    // invariant: s[0..h-1] is reverse of s[k+1..]
    while (h < k) {
        if (s.charAt(h) != s.charAt(k))
            return false;
        h= h+1; k= k-1;
    }
    return true;
}
```

Loop invariant — invariant because it’s true before/after each loop iteration

These sections are each others’ reverse
Break a project up into parts, making them as independent as possible. When the parts are constructed, put them together.

Each part can be understood by itself, without mentioning the others.
Algorithm to compute $b^c$.

Can’t understand any piece of it without understanding all. In fact, only way to get a handle on it is to execute it on some test case.

Need to understand initialization without looking at any other code.
Need to understand condition $y 
eq 0$ without looking at loop body
Etc.
Invariant: is true before and after each iteration

```
initialization;
// invariant P
while (B) {S}
```

Upon termination, we know P true, B false

“invariant” means unchanging. **Loop invariant**: an assertion—a true-false statement—that is true before and after each iteration of the loop—every time B is to be evaluated.

Help us understand each part of loop without looking at all other parts.
Simple example to illustrate methodology

Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv:  s = sum of 0..k-1 &&
//      0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of 0..n}

First loopy question.
Does it start right?
Does initialization make invariant true?

Yes!

s = sum of 0..k-1
= <substitute initialization>
  0 = sum of 0..1-1
= <arithmetic>
  0 = sum of 0..0

We understand initialization without looking at any other code
Simple example to illustrate methodology

Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv: s = sum of 0..k-1 &&
// 0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of 0..n}

We understand that postcondition is true without looking at init or repetend

Second loopy question. Does it stop right?
Upon termination, is postcondition true?

Yes!

inv && ! k <= n
=> <look at inv>
inv && k = n+1
=> <use inv>
s = sum of 0..n+1-1
Simple example to illustrate methodology

Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
\(k = 1; s = 0;
// inv: s = \text{sum of } 0..k-1 \quad \&\& \quad 0 \leq k \leq n+1
\)
while (k <= n) {
  s = s + k;
  k = k + 1;
}
{s = \text{sum of } 0..n}

Third loopy question.
Progress?
Does the repetend make progress toward termination?

Yes! Each iteration increases k, and when it gets larger than n, the loop terminates.

We understand that there is no infinite looping without looking at init and focusing on ONE part of the repetend.
Simple example to illustrate methodology

Store sum of 0..n in s
Precondition: n >= 0
// { n >= 0}
k= 1; s= 0;
// inv: s = sum of 0..k-1 &&
// 0 <= k <= n+1
while (k <= n) {
    s= s + k;
    k= k + 1;
}
{s = sum of 0..n}

Fourth loopy question.
Invariant maintained by each iteration?
Is this property true?
    {inv && k <= n} repetend {inv}
    Yes!

{s = sum of 0..k-1}
s= s + k;
{s = sum of 0..k}
k= k+1;
{s = sum of 0..k-1}
4 loopy questions to ensure loop correctness

{precondition Q}
init;
// invariant P
while (B) {
   S
}
{R}

First loopy question;
Does it start right?
Is {Q} init {P} true?

Second loopy question:
Does it stop right?
Does P && ! B imply R?

Third loopy question:
Does repetend make progress?
Will B eventually become false?

Fourth loopy question:
Does repetend keep invariant true?
Is {P && ! B} S {P} true?

Four loopy questions: if answered yes, algorithm is correct.
Note on ranges m..n

Range \( m..n \) contains \( n+1-m \) ints: \( m, m+1, ..., n \)
(Think about this as “Follower \((n+1)\) minus First \((m)\)”)

2..4 contains 2, 3, 4: that is \( 4 + 1 - 2 = 3 \) values

2..3 contains 2, 3: that is \( 3 + 1 - 2 = 2 \) values

2..2 contains 2: that is \( 2 + 1 - 2 = 1 \) value

2..1 contains: that is \( 1 + 1 - 2 = 0 \) values

Convention: notation \( m..n \) implies that \( m \leq n+1 \)

Assume convention even if it is not mentioned!

If \( m \) is 1 larger than \( n \), the range has 0 values

array segment \( b[m..n] \):

\[
\begin{array}{c|c|c}
\hline
m & & n \\
\hline
\end{array}
\]

b
Can’t understand this example without invariant!

Given \( c \geq 0 \), store \( b^c \) in \( z \)

\[
\begin{align*}
z & = 1; \quad x = b; \quad y = c; \\
\text{// invariant } & \quad y \geq 0 \quad \&\& \\
\text{// } & \quad z \cdot x^y = b^c \\
\text{while } & \quad (y \neq 0) \quad \{ \\
\text{if } & \quad (y \text{ is even}) \quad \{ \\
\text{ } & \quad x = x \cdot x; \quad y = y / 2; \\
\text{ } & \quad \} \quad \text{else } \{ \\
\text{ } & \quad z = z \cdot x; \quad y = y - 1; \\
\text{ } & \quad \} \\
\} \\
\{ & \quad z = b^c \}
\end{align*}
\]

First loopy question.
Does it start right?
Does initialization make invariant true?

Yes!

\[
\begin{align*}
z \cdot x^y \\
= & \quad <\text{substitute initialization}> \\
= & \quad 1 \cdot b^c \\
= & \quad <\text{arithmetic}> \\
= & \quad b^c
\end{align*}
\]

We understand initialization without looking at any other code
For loopy questions to reason about invariant

Given $c \geq 0$, store $b^c$ in $x$

```
z = 1; x = b; y = c;
// invariant $y \geq 0$ AND
// $z \times x^y = b^c$
while (y != 0) {
    if (y is even) {
        x = x \times x; y = y / 2;
    } else {
        z = z \times x; y = y - 1;
    }
}
{z = b^c}
```

Second loopy question.
Does it stop right?
When loop terminates, is $z = b^c$?

Yes! Take the invariant, which is true, and use fact that $y = 0$:

```
z \times x^y = b^c
= <y = 0>
z \times x^0 = b^c
= <arithmetic>
z = b^c
```

We understand loop condition without looking at any other code
Given $c \geq 0$, store $b^c$ in $x$

$z=1; \ x=b; \ y=c;$

// invariant $y \geq 0$ AND
// $z \times x^y = b^c$

while ($y \neq 0$) {
    if ($y$ is even) {
        $x=x \times x; \ y=y/2;$
    } else {
        $z=z \times x; \ y=y-1;$
    }
}

$\{z = b^c\}$

Third loopy question.  
Does repetend make progress toward termination?

Yes! We know that $y > 0$ when loop body is executed. The loop body decreases $y$.

We understand progress without looking at initialization.
For loopy questions to reason about invariant

Given $c \geq 0$, store $b^c$ in $x$

```java
z = 1; x = b; y = c;
// invariant $y \geq 0$ AND
// $z \times x^y = b^c$
while (y != 0) {
    if (y is even) {
        x = x*x; y = y/2;
    } else {
        z = z*x; y = y - 1;
    }
}
{z = b^c}
```

Fourth loopy question. Does repetend keep invariant true?

Yes! Because of properties:

- For $y$ even, $x^y = (x^2)^{y/2}$
- $z \times x^y = z \times x \times x^{y-1}$

We understand invariance without looking at initialization
Develop binary search for v in sorted array b

Example:

If v is 4, 5, or 6, h is 5
If v is 7 or 8, h is 6

If v in b, h is index of rightmost occurrence of v.
If v not in b, h is index before where it belongs.
Develop binary search in sorted array \( b \) for \( v \)

\[
\begin{array}{cc}
0 & b.\text{length} \\
\text{pre:} & b \quad \quad ?
\end{array}
\]

Store a value in \( h \) to make this true:

\[
\begin{array}{ccc}
0 & h & b.\text{length} \\
\text{post:} & b \quad \quad \leq v & > v
\end{array}
\]

Get loop invariant by combining pre- and post-conditions, adding variable \( t \) to mark the other boundary

\[
\begin{array}{cccc}
0 & h & t & b.\text{length} \\
\text{inv:} & b \quad \quad \leq v & ? & > v
\end{array}
\]
How does it start (what makes the invariant true)?

pre: 

| 0 | b | ? | b.length |

inv: 

| 0 | 0 | h | t | b.length |

| <= v | ? | > v |

Make first and last partitions empty:

h = -1; t = b.length;
When does it end (when does invariant look like postcondition)?

\[
\begin{array}{cccc}
<table>
<thead>
<tr>
<th>\text{inv:}</th>
<th>\text{post:}</th>
</tr>
</thead>
</table>
| b | \begin{array}{c|c|c}
0 & h & b.\text{length} \\
\leq v & > v \\
\end{array}|
\end{array}
\]

\[
\begin{array}{cccc}
<table>
<thead>
<tr>
<th>\text{inv:}</th>
<th>\text{post:}</th>
</tr>
</thead>
</table>
| b | \begin{array}{c|c|c|c}
0 & h & t & b.\text{length} \\
\leq v & ? & > v \\
\end{array}|
\end{array}
\]

\[h = -1; \ t = b.\text{length};\]

\[\textbf{while} ( h \neq t-1 ) \{\]

\[\}

Stop when \( ? \) section is empty. That is when \( h = t-1 \).

Therefore, continue as long as \( h \neq t-1 \).
How does body make progress toward termination (cut in half) and keep invariant true?

Let $e$ be index of middle value of $?$ section.
Maybe we can set $h$ or $t$ to $e$, cutting $?$ section in half.
How does body make progress toward termination (cut ? in half) and keep invariant true?

```java
h= -1;  t= b.length;
while ( h != t-1 ) {
    int e= (h+t)/2;
    if (b[e] <= v) h= e;
}
```

If \( b[e] \leq v \), then so is every value to its left, since the array is sorted. Therefore, \( h= e \); keeps the invariant true.
How does body make progress toward termination (cut ? in half) and keep invariant true?

inv: 

If \( b[e] > v \), then so is every value to its right, since the array is sorted. Therefore, \( t = e \); keeps the invariant true.
Develop binary search in sorted array b for v

<table>
<thead>
<tr>
<th>0</th>
<th>b</th>
<th>h</th>
<th>b.length</th>
</tr>
</thead>
</table>

**pre:**  
Store a value in h to make this true:

<table>
<thead>
<tr>
<th>0</th>
<th>h</th>
<th>b.length</th>
</tr>
</thead>
</table>

**post:**  

| b | <= v | > v |

**DON'T TRY TO MEMORIZE CODE!**

Instead, learn to derive the loop invariant from the pre- and post-condition and then to develop the loop using the pre- and post-condition and the loop invariant.

**PRACTICE THIS ON KNOWN ALGORITHMS!**
Processing arrays from beg to end (or end to beg)

Many loops process elements of an array $b$ (or a String, or any list) in order: $b[0]$, $b[1]$, $b[2]$, …

If the postcondition is

$$R: \ b[0..b.length-1] \ \text{has been processed}$$

Then in the beginning, nothing has been processed, i.e.

$$b[0..-1] \ \text{has been processed}$$

After $k$ iterations, $k$ elements have been processed:

$$P: \ b[0..k-1] \ \text{has been processed}$$

$$invariant \ P: \begin{array}{|c|c|c|} \hline 0 & \text{processed} & b.length \ \text{not processed} \ \hline \end{array}$$
Processing arrays from beg to end (or end to beg)

Task: Process $b[0..b.length-1]$  

Replace $b.length$ in postcondition by fresh variable $k$ to get invariant  

$b[0..k-1]$ has been processed  

or draw it as a picture

```plaintext
k = 0;
{inv P}
while (k != b.length) {
    Process $b[k]$; // maintain invariant
    k = k + 1; // progress toward termination
}
{R: $b[0..b.length-1]$ has been processed}
```

inv P:  

<table>
<thead>
<tr>
<th>0</th>
<th>k</th>
<th>b.length</th>
</tr>
</thead>
<tbody>
<tr>
<td>processed</td>
<td>not processed</td>
<td></td>
</tr>
</tbody>
</table>
Processing arrays from beg to end (or end to beg)

Task: Process $b[0..b.length-1]$

```java
k = 0;
{inv P}
while (k != b.length) {
    Process b[k]; // maintain invariant
    k = k + 1; // progress toward termination
}
{R: $b[0..b.length-1]$ has been processed}
```

Most loops that process the elements of an array in order will have this loop invariant and will look like this.

<table>
<thead>
<tr>
<th>0</th>
<th>k</th>
<th>$b.length$</th>
</tr>
</thead>
<tbody>
<tr>
<td>processed</td>
<td>not processed</td>
<td></td>
</tr>
</tbody>
</table>

inv P: $b$
Count the number of zeros in b.
Start with last program and refine it for this task

Task: Set s to the number of 0’s in \( b[0..b.length-1] \)

\[
\begin{align*}
\text{k} &= 0; \\
\text{s} &= 0; \\
\{\text{inv } P\} \\
\text{while} \ (\text{k} \neq b.\text{length}) \quad \{ \\
\quad \text{if} \ (b[k] == 0) \quad \text{s} = \text{s} + 1; \\
\quad \text{k} = \text{k} + 1; & \quad \text{// progress toward termination} \\
\} \\
\{\text{R: s = number of 0’s in } b[0..b.length-1]\}
\end{align*}
\]

\[
\begin{array}{cccc}
0 & k & \text{b.length} \\
\text{inv P: } b & \text{s = # 0’s here} & \text{not processed}
\end{array}
\]
Be careful. Invariant may require processing elements in reverse order!

This invariant forces processing from beginning to end

\[
\begin{array}{ccc}
0 & k & b.\text{length} \\
\end{array}
\]

\text{inv P: } \begin{array}{cc}
b & \text{processed} & \text{not processed} \\
\end{array}

This invariant forces processing from end to beginning

\[
\begin{array}{ccc}
0 & k & b.\text{length} \\
\end{array}
\]

\text{inv P: } \begin{array}{cc}
b & \text{not processed} & \text{processed} \\
\end{array}
Process elements from end to beginning

\[ k = b.length - 1; \quad \text{// how does it start?} \]

\[
\text{while (} (k \geq 0) \text{) } \{
\begin{align*}
\text{ Process } b[k]; & \quad \text{// how does it maintain invariant?} \\
\text{ k } &= \text{ k } - 1; & \quad \text{// how does it make progress?}
\end{align*}
\}
\]

\{R: b[0..b.length-1] is processed\}

\begin{array}{c|c|c}
0 & k & b.length \\
\hline
\text{inv P: } b & \text{not processed} & \text{processed}
\end{array}
Process elements from end to beginning

k = b.length - 1;
while (k >= 0) {
    Process b[k];
k = k – 1;
}

{R: b[0..b.length-1] is processed}

Heads up! It is important that you can look at an invariant and decide whether elements are processed from beginning to end or end to beginning!

For some reason, some students have difficulty with this. A question like this could be on the prelim!