Overview references to sections in text

- Note: We’ve covered everything in JavaSummary.pptx!
- What is recursion? 7.1-7.39 slide 1-7
- Base case 7.1-7.10 slide 13
- How Java stack frames work 7.8-7.10 slide 28-32
Function equals

```java
public class Object {
    /** Return true iff this object is
     * the same as ob */
    public boolean equals(Object b) {
        return this == b;
    }
}
```

This gives a null-pointer exception:

```java
null.equals(y)
```

x.equals(y) is same as x == y except when x is null!
Override function `equals` in a class to give meaning to:

“these two (possibly different) objects of the class have the same values in some of their fields”

For those who are mathematically inclined, like any equality function, `equals` should be reflexive, symmetric, and transitive.

**Reflexive**: `b.equals(b)`

**Symmetric**: `b.equals(c) = c.equals(b)`

**Transitive**: if `b.equals(c)` and `c.equals(d)`, then `b.equals(d)`
public class Animal {
/** = “h is an Animal with the same values in its fields as this Animal” */
public boolean equals (Object h) {
    if (!(h instanceof Animal))
        return false;
    Animal ob = (Animal) h;
    return name.equals(ob.name) && age == ob.age;
}

1. Because of h is an Animal in spec, need the test h instanceof Animal
2. In order to be able to reference fields in partition `Animal`, need to cast `h` to `Animal`
public class Animal {
    /** = “h is an Animal with the same values in its fields as this Animal” */
    public boolean equals (Object h) {
        if (!(h instanceof Animal))
            return false;
        Animal ob = (Animal) h;
        return name.equals(ob.name) &&
            age == ob.age;
    }
}

3. Use String equals function to check for equality of String values. Use == for primitive types
Why can’t the parameter type be Animal?

```java
public class Animal {
    /** = “h is an Animal with the same values in its fields as this Animal” */
    public boolean equals (Animal h) {
        if (!(h instanceof Animal))
            return false;
        Animal ob = (Animal) h;
        return name.equals(ob.name) &&
               age == ob.age;
    }
}
```

What is wrong with this?
Sum the digits in a non-negative integer

```java
/** return sum of digits in n.
 * Precondition: n >= 0 */

public static int sum(int n) {
    if (n < 10) return n;
    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10) + n%10;
}
```

E.g. sum(7) = 7

E.g. sum(8703) = sum(870) + 3;
Two issues with recursion

1. Why does it work? How does the method executed?

```java
/** return sum of digits in n.
 * Precondition: n >= 0 */

public static int sum(int n) {
    if (n < 10) return n;
    // { n has at least two digits }
    // return first digit + sum of rest
    return sum(n/10) + n%10 + ;
}
```

sum calls itself!

2. How do we understand a given recursive method or how do we write/develop a recursive method?
Stacks and Queues

Stack: list with (at least) two basic ops:
* Push an element onto its top
* Pop (remove) top element

Last-In-First-Out (LIFO)

Like a stack of trays in a cafeteria

Queue: list with (at least) two basic ops:
* Append an element
* Remove first element

First-In-First-Out (FIFO)

Americans wait in a line the Brits wait in a queue!
Stack Frame

A “frame” contains information about a method call:

At runtime Java maintains a stack that contains frames for all method calls that are being executed but have not completed.

Method call: push a frame for call on stack assign argument values to parameters execute method body. Use the frame for the call to reference local variables parameters.

End of method call: pop its frame from the stack; if it is a function leave the return value on top of stack.
```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

Frame for method in the system that calls method main
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(
    String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

Frame for method in the system that calls method main: main is then called
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

Method main calls sum:
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

$n \geq 10$ sum calls sum:
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

$n \geq 10$. sum calls sum:
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

n < 10 sum stops: frame is popped and n is put on stack:
Example: Sum the digits in a non-negative integer

```
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

Using return value 8 stack computes 8 + 2 = 10 pops frame from stack puts return value 10 on stack
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

Using return value 10 stack computes
10 + 4 = 14 pops frame from stack
puts return value 14 on stack
Example: Sum the digits in a non-negative integer

```java
public static int sum(int n) {
    if (n < 10) return n;
    return sum(n/10) + n%10;
}

public static void main(
    String[] args) {
    int r = sum(824);
    System.out.println(r);
}
```

Using return value 14 main stores 14 in r and removes 14 from stack.
Summary of method call execution

Memorize this!

1. A frame for a call contains parameters, local variables, and other information needed to properly execute a method call.

2. To execute a method call: push a frame for the call on the stack, assign values to parameters, execute method body, pop frame for call from stack, and (for a function) push returned value on stack.

When executing method body look in frame for call for parameters and local variables.
Questions about local variables

In a call m()
when is local variable d created and when is it destroyed?
Which version of procedure m do you like better? Why?

```java
public static void m(...) {
    ...
    while (...) {
        int d = 5;
        ...
    }
}
```

```java
public static void m(...) {
    int d;
    ...
    while (...) {
        d = 5;
        ...
    }
}
```
Recursion is used extensively in math

Math definition of n factorial

\[ 0! = 1 \]
\[ n! = n \times (n-1)! \quad \text{for } n > 0 \]

E.g. \( 3! = 3 \times 2 \times 1 = 6 \)

Math definition of \( b^c \) for \( c \geq 0 \)

\[ b^0 = 1 \]
\[ b^c = b \times b^{c-1} \quad \text{for } c > 0 \]

Easy to make math definition into a Java function!

```java
public static int fact(int n) {
    if (n == 0) return 1;
    return n \times \text{fact}(n-1);\}
```

Lots of things defined recursively:
expression grammars trees ….
We will see such things later
Two views of recursive methods

- How are calls on recursive methods executed? We saw that. Use this only to gain understanding / assurance that recursion works.

- How do we understand a recursive method — know that it satisfies its specification? How do we write a recursive method? This requires a totally different approach. Thinking about how the method gets executed will confuse you completely! We now introduce this approach.
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s): Cases where the parameter is small enough that the result can be computed simply and without recursive calls.

If \( n < 10 \) then \( n \) consists of a single digit. Looking at the spec we see that that digit is the required sum.

```java
/** = sum of digits of \( n \).
 * Precondition: \( n >= 0 \) */
public static int sum(int n) {
    if (n < 10) return n;
    // \( n \) has at least two digits
    return sum(n/10) + n%10;
}
```
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the method spec and verify that the correct result is then obtained.

```java
/** = sum of digits of n. * Precondition:  n >= 0 */
public static int sum(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return sum(n/10) + n%10;
}
```

return sum(n/10) + n%10;

return (sum of digits of n/10) + n%10;  // e.g. n = 843
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does acc. to the spec and verify correctness.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

```java
/** = sum of digits of n. *
 * Precondition: n >= 0 */
 public static int sum(int n) {
    if (n < 10) return n;
    // n has at least two digits
    return sum(n/10) + n%10;
}
```

\[ n/10 < n \]
Understanding a recursive method

Step 1. Have a precise spec!

Step 2. Check that the method works in the base case(s).

Step 3. Look at the recursive case(s). In your mind replace each recursive call by what it does according to the spec and verify correctness.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.

Important! Can’t do step 3 without it

Once you get the hang of it this is what makes recursion easy! This way of thinking is based on math induction which we will see later in the course.
Step 1. Have a precise spec!

Step 2. Write the base case(s): Cases in which no recursive calls are needed. Generally for “small” values of the parameters.

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind. Done suitably point 4 is automatically satisfied.

Step 4. (No infinite recursion) Make sure that the args of recursive calls are in some sense smaller than the pars of the method.
For the rest of the class we demo writing recursive functions using the approach outlined below. The java file we develop will be placed on the course webpage some time after the lecture.

Step 1. Have a precise spec!

Step 2. Write the base case(s).

Step 3. Look at all other cases. See how to define these cases in terms of smaller problems of the same kind. Then implement those definitions using recursive calls for those smaller problems of the same kind.
The Fibonacci Function

Mathematical definition:
\[ \text{fib}(0) = 0 \]
\[ \text{fib}(1) = 1 \]
\[ \text{fib}(n) = \text{fib}(n - 1) + \text{fib}(n - 2) \quad n \geq 2 \]

Fibonacci sequence: 0 1 1 2 3 5 8 13 ...

```c
/** = fibonacci(n). Pre: n >= 0 */
static int fib(int n) {
    if (n <= 1) return n;
    // { 1 < n }
    return fib(n-2) + fib(n-1);
}
```

Fibonacci (Leonardo Pisano) 1170-1240?

Statue in Pisa Italy
Giovanni Paganucci 1863
A String palindrome is a String that reads the same backward and forward.

A String with at least two characters is a palindrome if

- (0) its first and last characters are equal and
- (1) chars between first & last form a palindrome:

```
A recursive definition!
```

```e.g. AMANAPLANACANALPANAMA
```

```have to be the same
```

```have to be a palindrome
```
Example: Is a string a palindrome?

```java
/** = "s is a palindrome" */
public static boolean isPal(String s) {
    if (s.length() <= 1)
        return true;
    // { s has at least 2 chars }
    int n = s.length()-1;
    return s.charAt(0) == s.charAt(n) && isPal(s.substring(1, n));
}
```

isPal(“racecar”) returns true
isPal(“pumpkin”) returns false
A man a plan a caret a ban a myriad a sum a lac a liar a hoop a pint a catalpa a gas an oil a bird a yell a vat a caw a pax a wag a tax a nay a ram a cap a yam a gay a tsar a wall a car a luger a ward a bin a woman a vassal a wolf a tuna a nit a pall a fret a watt a bay a daub a tan a cab a datum a gall a hat a fag a zap a say a jaw a lay a wet a gallop a tug a trot a trap a tram a torr a caper a top a tonk a toll a ball a fair a sax a minim a tenor a bass a passer a capital a rut an amen a ted a cabal a tang a sun an ass a maw a sag a jam a dam a sub a salt an axon a sail an ad a wadi a radian a room a rood a rip a tad a pariah a revel a reel a reed a pool a plug a pin a peek a parabola a dog a pat a cud a nu a fan a pal a rum a nod an eta a lag an eel a batik a mug a mot a nap a maxim a mood a leek a grub a gob a gel a drab a citadel a total a cedar a tap a gag a rat a manor a bar a gal a cola a pap a yaw a tab a raj a gab a nag a pagan a bag a jar a bat a way a papa a local a gar a baron a mat a rag a gap a tar a decal a tot a led a tic a bard a leg a bog a burg a keel a doom a mix a map an atom a gum a kit a baleen a gala a ten a don a mural a pan a faun a ducat a pagoda a lob a rap a keep a nip a gulp a loop a deer a leer a lever a hair a pad a tapir a door a moor an aid a raid a wad an alias an ox an atlas a bus a madam a jag a saw a mass an anus a gnat a lab a cadet an em a natural a tip a caress a pass a baronet a minimax a sari a fall a ballot a knot a pot a rep a carrot a mart a part a tort a gut a poll a gateway a law a jay a sap a zag a fat a hall a gamut a dab a can a tabu a day a batt a waterfall a patina a nut a flow a lass a van a mow a nib a draw a regular a call a war a stay a gam a yap a cam a ray an ax a tag a wax a paw a cat a valley a drib a lion a saga a plat a catnip a pooh a rail a calamus a dairyman a bater a canal Panama
Example: Count the e’s in a string

```java
/** = number of times c occurs in s */
public static int countEm(char c String s) {
    if (s.length() == 0) return 0;
    // { s has at least 1 character }
    if (s.charAt(0) != c)
        return countEm(c s.substring(1));
    // { first character of s is c}
    return 1 + countEm (c s.substring(1));
}
```

- countEm('e' “it is easy to see that this has many e’s”) = 4
- countEm('e' “Mississippi”) = 0