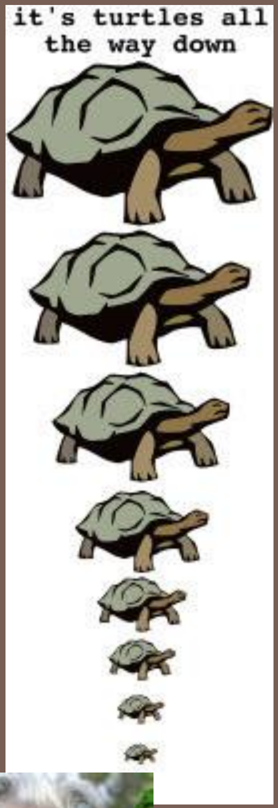


A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the center of a vast collection of stars called our galaxy.

At the end of the lecture, a little old lady at the back of the room got up and said: "What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise." The scientist gave a superior smile before replying, "What is the tortoise standing on?" "You're very clever, young man, very clever", said the old lady. "But it's turtles all the way down!"

INDUCTION

Lecture 20

CS2110 – Fall 2013

Overview: Reasoning about Programs

- Our broad problem: code is unlikely to be correct if we don't have good reasons for believing it works
  - We need clear problem statements
  - And then a rigorous way to convince ourselves that what we wrote solves the problem
- But reasoning about programs can be hard
  - Especially with recursion, concurrency
  - Today focus on recursion

Overview: Reasoning about Programs

- Recursion
  - A programming strategy that solves a problem by reducing it to simpler or smaller instance(s) of the same problem
- Induction
  - A mathematical strategy for proving statements about natural numbers 0,1,2,... (or more generally, about inductively defined objects)
- They are very closely related

- Induction can be used to establish the *correctness* and *complexity* of programs

Defining Functions

- It is often useful to describe a function in different ways
  - Let $S : \text{int} \rightarrow \text{int}$ be the function where S(n) is the sum of the integers from 0 to n. For example,

    $S(0) = 0 \qquad\qquad S(3) = 0+1+2+3 = 6$

  - Definition: iterative form
    - $S(n) = 0+1+ \ldots + n$

      $= \sum_{i=0}^{n} i$

  - Another characterization: closed form
    - $S(n) = n(n+1)/2$

Sum of Squares

- A more complex example
  - Let SQ : int $\longrightarrow$ int be the function that gives the sum of the squares of integers from 0 to n:

$$SQ(0) = 0$$

$$SQ(3) = 0^2 + 1^2 + 2^2 + 3^2 = 14$$

- Definition (iterative form):

$$SQ(n) = 0^2 + 1^2 + \ldots + n^2$$

- Is there an equivalent closed-form expression?

Closed-Form Expression for SQ(n)

- Sum of integers between 0 through n was n(n+1)/2 which is a *quadratic* in n (that is, $O(n^2)$)

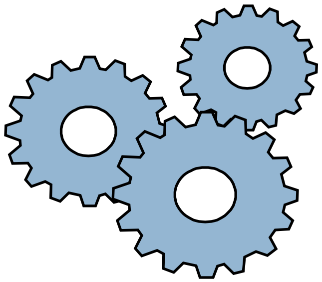
- Inspired guess: perhaps sum of *squares* of integers between 0 through n is a *cubic* in n

- Conjecture: $SQ(n) = an^3+bn^2+cn+d$
  where a, b, c, d are unknown coefficients

- How can we find the values of the four unknowns?
  - Idea: Use any 4 values of n to generate 4 linear equations, and then solve

Finding Coefficients

$$SQ(n) = 0^2+1^2+\ldots+n^2 = an^3+bn^2+cn+d$$



- Use $n = 0, 1, 2, 3$
  - $SQ(0) = 0 = a\cdot 0 + b\cdot 0 + c\cdot 0 + d$
  - $SQ(1) = 1 = a\cdot 1 + b\cdot 1 + c\cdot 1 + d$
  - $SQ(2) = 5 = a\cdot 8 + b\cdot 4 + c\cdot 2 + d$
  - $SQ(3) = 14 = a\cdot 27 + b\cdot 9 + c\cdot 3 + d$
  - Solve these 4 equations to get
    - $a = 1/3$ $b = 1/2$ $c = 1/6$ $d = 0$

Is the Formula Correct?

- This suggests

$$
\begin{aligned}
SQ(n) &= 0^2 + 1^2 + \ldots + n^2 \\
&= n^3/3 + n^2/2 + n/6 \\
&= n(n+1)(2n+1)/6
\end{aligned}
$$

- Question: Is this closed-form solution true for all n?
  - Remember, we only used n = 0,1,2,3 to determine these coefficients
  - We do not know that the closed-form expression is valid for other values of n

One Approach

- Try a few other values of n to see if they work.
  - Try n = 5: SQ(n) = 0+1+4+9+16+25 = 55
  - Closed-form expression: 5·6·11/6 = 55
  - Works!

- Try some more values…

- We can never prove validity of the closed-form solution for all values of n this way, since there are an infinite number of values of n

A Recursive Definition

- To solve this problem, let's express SQ(n) in a different way:
  - $SQ(n) = \boxed{0^2 + 1^2 + \ldots + (n-1)^2} + n^2$
  - The part in the box is just SQ(n-1)

- This leads to the following *recursive* definition
  - $SQ(0) = 0$ ← Base Case
  - $SQ(n) = SQ(n-1) + n^2, \ n > 0$ ← Recursive Case

- Thus,
  - $SQ(4) = SQ(3) + 4^2 = SQ(2) + 3^2 + 4^2 = SQ(1) + 2^2 + 3^2 + 4^2 = SQ(0) + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1^2 + 2^2 + 3^2 + 4^2$

Are These Two Functions Equal?

- SQ_r (r = recursive)

$$SQ_r(0) = 0$$
$$SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$$

- SQ_c (c = closed-form)

$$SQ_c(n) = n(n+1)(2n+1)/6$$

Induction over Integers

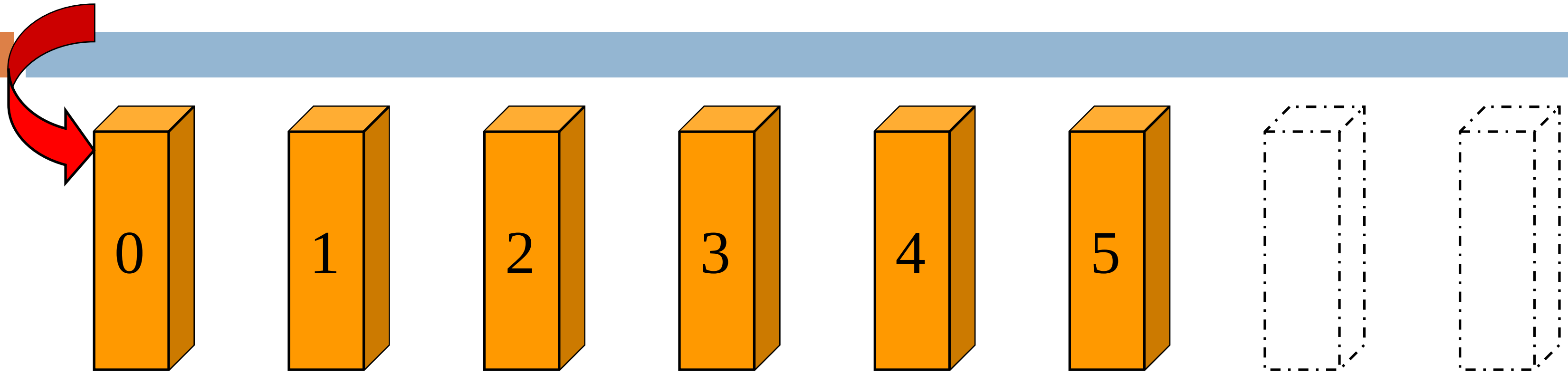
- To prove that some property P(n) holds for all integers $n \geq 0$,

1. **Basis:** Show that P(0) is true

2. **Induction Step:** Assuming that P(k) is true for an *unspecified* integer k, show that P(k+1) is true

- **Conclusion:** Because we could have picked any k, we conclude that P(n) holds for all integers $n \geq 0$

Dominos



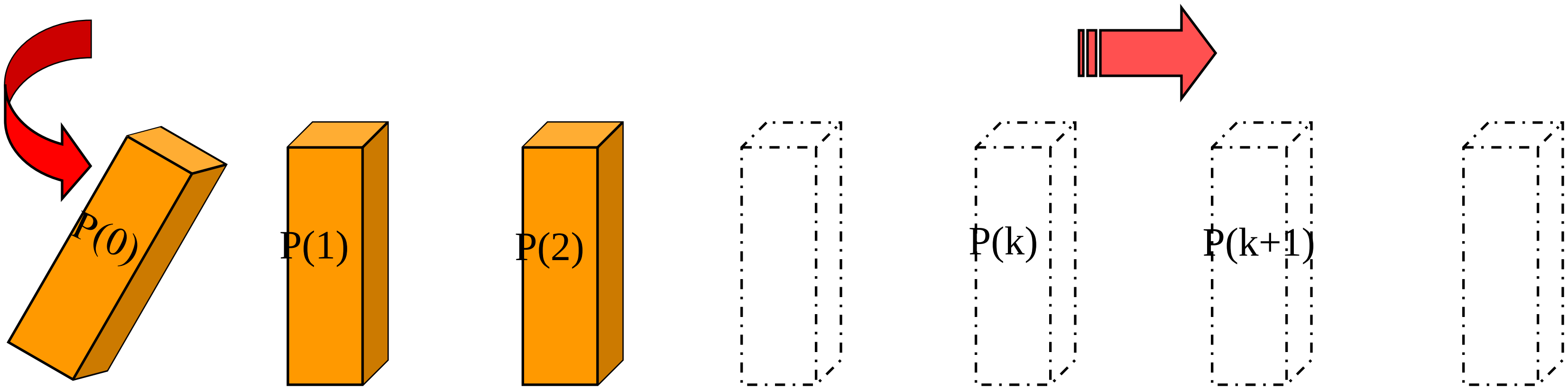
- Assume equally spaced dominos, and assume that spacing between dominos is less than domino length
- How would you argue that all dominos would fall?
- Dumb argument:
  - Domino 0 falls because we push it over
  - Domino 0 hits domino 1, therefore domino 1 falls
  - Domino 1 hits domino 2, therefore domino 2 falls
  - Domino 2 hits domino 3, therefore domino 3 falls
  - …
- Is there a more compact argument we can make?

Better Argument

- Argument:
  - Domino 0 falls because we push it over (Base Case or Basis)
  - Assume that domino k falls over (Induction Hypothesis)
  - Because domino k's length is larger than inter-domino spacing, it will knock over domino k+1 (Inductive Step)
  - Because we could have picked any domino to be the k^{th} one, we conclude that all dominos will fall over (Conclusion)
- This is an inductive argument
- This version is called *weak induction*
  - There is also *strong induction* (later)
- Not only is this argument more compact, it works for an arbitrary number of dominoes!

$SQ_r(n) = SQ_c(n)$ for all n?

- Define P(n) as $SQ_r(n) = SQ_c(n)$



- Prove P(0)
- Assume P(k) for unspecified k, and then prove P(k+1) under this assumption

Proof (by Induction)

- Recall: $SQ_r(0) = 0$
  $SQ_r(n) = SQ_r(n-1) + n^2, \quad n > 0$

  $SQ_c(n) = n(n+1)(2n+1)/6$
- Let P(n) be the proposition that $SQ_r(n) = SQ_c(n)$
- Basis: P(0) holds because $SQ_r(0) = 0$ and $SQ_c(0) = 0$ by definition
- Induction Hypothesis: Assume $SQ_r(k) = SQ_c(k)$
- Inductive Step:

$SQ_r(k+1)$	$= SQ_r(k) + (k+1)^2$	by definition of $SQ_r(k+1)$
	$= SQ_c(k) + (k+1)^2$	by the Induction Hypothesis
	$= k(k+1)(2k+1)/6 + (k+1)^2$	by definition of $SQ_c(k)$
	$= (k+1)(k+2)(2k+3)/6$	algebra
	$= SQ_c(k+1)$	by definition of $SQ_c(k+1)$

- Conclusion: $SQ_r(n) = SQ_c(n)$ for all $n \varepsilon 0$

Another Example

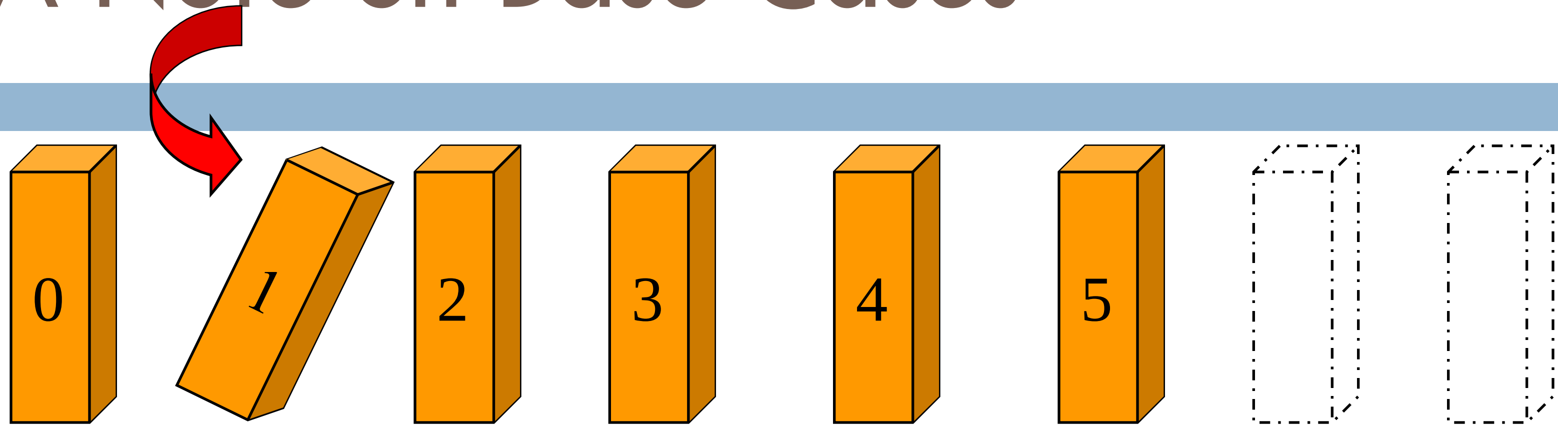
- Prove that $0+1+\ldots+n = n(n+1)/2$

- Basis: Obviously holds for $n = 0$
- Induction Hypothesis: Assume $0+1+\ldots+k = k(k+1)/2$
- Inductive Step:

$$
\begin{aligned}
0+1+\ldots+(k+1) &= [0+1+\ldots+k] + (k+1) && \text{by def} \\
&= k(k+1)/2 \;+\; (k+1) && \text{by I.H.} \\
&= (k+1)(k+2)/2 && \text{algebra}
\end{aligned}
$$

- Conclusion: $0+1+\ldots+n = n(n+1)/2$ for all $n \geq 0$

A Note on Base Cases



- Sometimes we are interested in showing some proposition is true for integers ≥ b
- Intuition: we knock over domino b, and dominoes in front get knocked over; not interested in 0,1,…,(b − 1)
- In general, the base case in induction does not have to be 0
- If base case is some integer b
  - Induction proves the proposition for n = b, b+1, b+2, …
  - Does not say anything about n = 0,1,…,b − 1

Weak Induction: Nonzero Base Case

- Claim: You can make any amount of postage above 8¢ with some combination of 3¢ and 5¢ stamps
- Basis: True for 8¢: $8 = 3 + 5$
- Induction Hypothesis: Suppose true for some $k \geq 8$
- Inductive Step:
  - If used a 5¢ stamp to make k, replace it by two 3¢ stamps. Get k+1.
  - If did not use a 5¢ stamp to make k, must have used at least three 3¢ stamps. Replace three 3¢ stamps by two 5¢ stamps. Get k+1.
- Conclusion: Any amount of postage above 8¢ can be made with some combination of 3¢ and 5¢ stamps

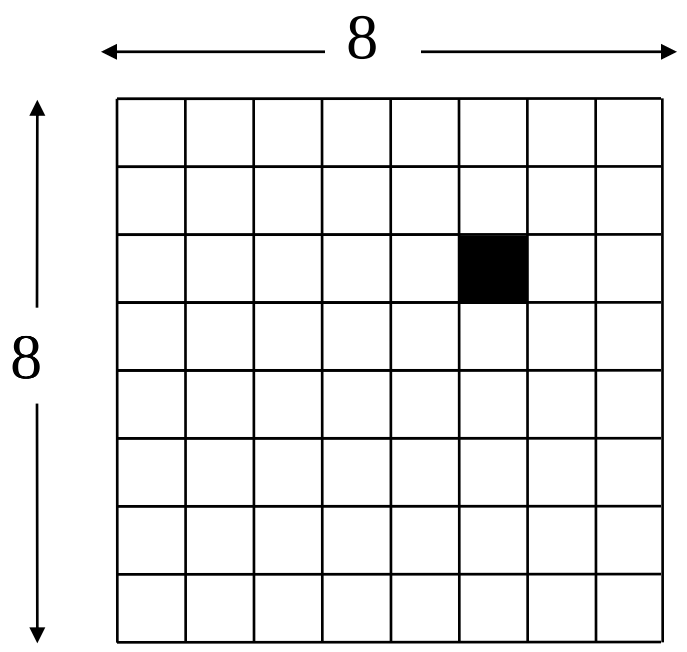
What are the "Dominos"?

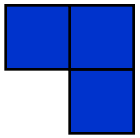
- In some problems, it can be tricky to determine how to set up the induction

- This is particularly true for geometric problems that can be attacked using induction

A Tiling Problem

- A chessboard has one square cut out of it
- Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Not obvious that we can use induction!



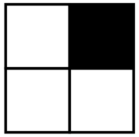


Proof Outline

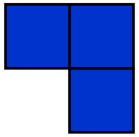
- Consider boards of size 2^n x 2^n for n = 1,2,…
- Basis: Show that tiling is possible for 2 x 2 board
- Induction Hypothesis: Assume the 2^k x 2^k board can be tiled
- Inductive Step: Using I.H. show that the 2^{k+1} x 2^{k+1} board can be tiled
- Conclusion: Any 2^n x 2^n board can be tiled, n = 1,2,…
  - Our chessboard (8 x 8) is a special case of this argument
  - We will have proven the 8 x 8 special case by solving a more general problem!

Basis

- The 2 x 2 board can be tiled regardless of which one of the four pieces has been omitted

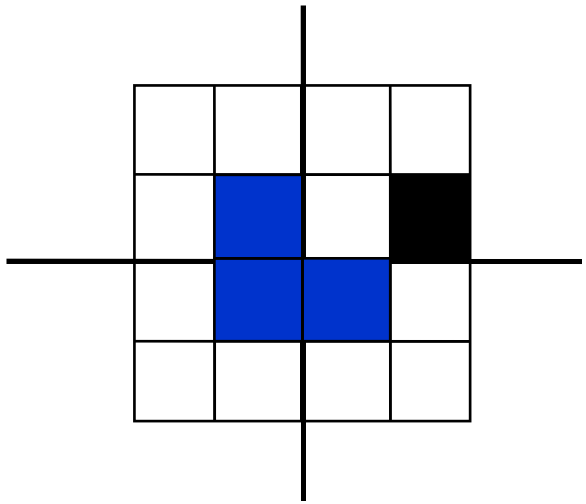


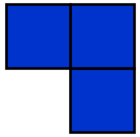
2 x 2 board



4 x 4 Case

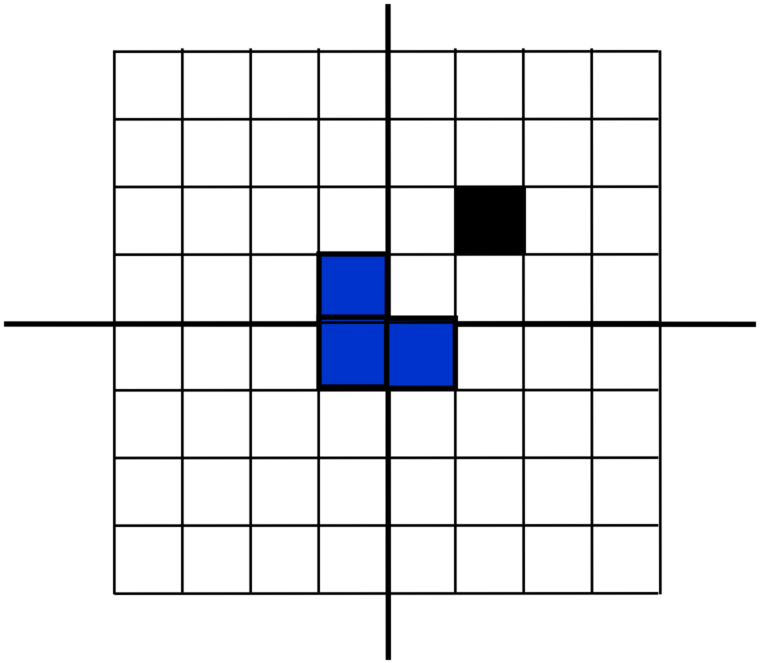
- Divide the 4 x 4 board into four 2 x 2 sub-boards
- One of the four sub-boards has the missing piece
  - By the I.H., that sub-board can be tiled since it is a 2 x 2 board with a missing piece
- Tile center squares of three remaining sub-boards as shown
  - This leaves three 2 x 2 boards, each with a missing piece
  - We know these can be tiled by the Induction Hypothesis

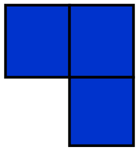




2^{k+1} x 2^{k+1} case

- Divide board into four sub-boards and tile the center squares of the three complete sub-boards
- The remaining portions of the sub-boards can be tiled by the I.H. (which assumes we can tile 2^k x 2^k boards)





When Induction Fails

- Sometimes an inductive proof strategy for some proposition may fail
- This does not necessarily mean that the proposition is wrong
  - It may just mean that the particular inductive strategy you are using is the wrong choice
- A different induction hypothesis (or a different proof strategy altogether) may succeed

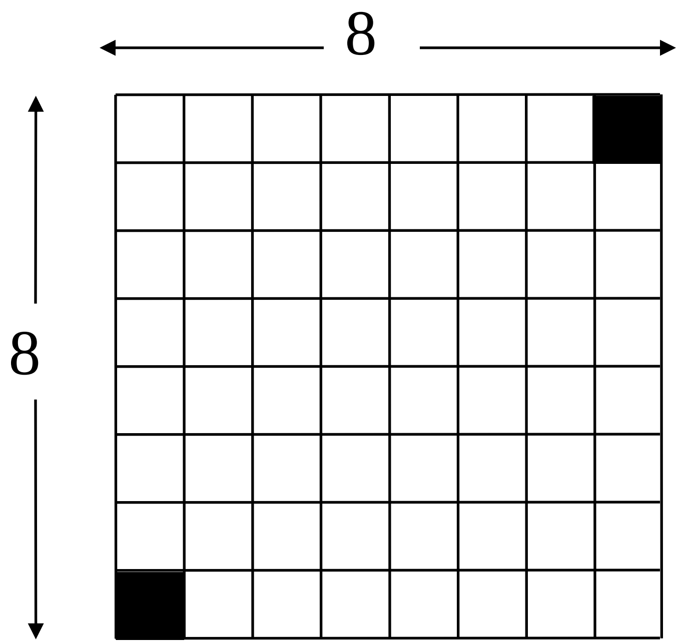
Tiling Example (Poor Strategy)

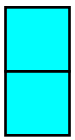
- Let's try a different induction strategy
- Proposition
  - Any n x n board with one missing square can be tiled
- Problem
  - A 3 x 3 board with one missing square has 8 remaining squares, but our tile has 3 squares; tiling is impossible
- Thus, any attempt to give an inductive proof of this proposition *must fail*

- Note that this failed proof does not tell us anything about the 8x8 case

A Seemingly Similar Tiling Problem

- A chessboard has opposite corners cut out of it. Can the remaining board be tiled using tiles of the shape shown in the picture (rotation allowed)?
- Induction fails here. Why? (Well…for one thing, this board can't be tiled with dominos.)





Strong Induction

- We want to prove that some property P holds for all n
- Weak induction
  - P(0): Show that property P is true for 0
  - P(k) ⇒ P(k+1): Show that if property P is true for k, it is true for k+1
  - Conclude that P(n) holds for all n
- Strong induction
  - P(0): Show that property P is true for 0
  - P(0) and P(1) and … and P(k) ⇒ P(k+1): show that if P is true for numbers less than or equal to k, it is true for k+1
  - Conclude that P(n) holds for all n
- Both proof techniques are equally powerful

Conclusion

- Induction is a powerful proof technique
- Recursion is a powerful programming technique
- Induction and recursion are closely related
  - We can use induction to prove correctness and complexity results about recursive programs