CS/ENGRD 2110
Object-Oriented Programming and Data Structures

Spring 2012

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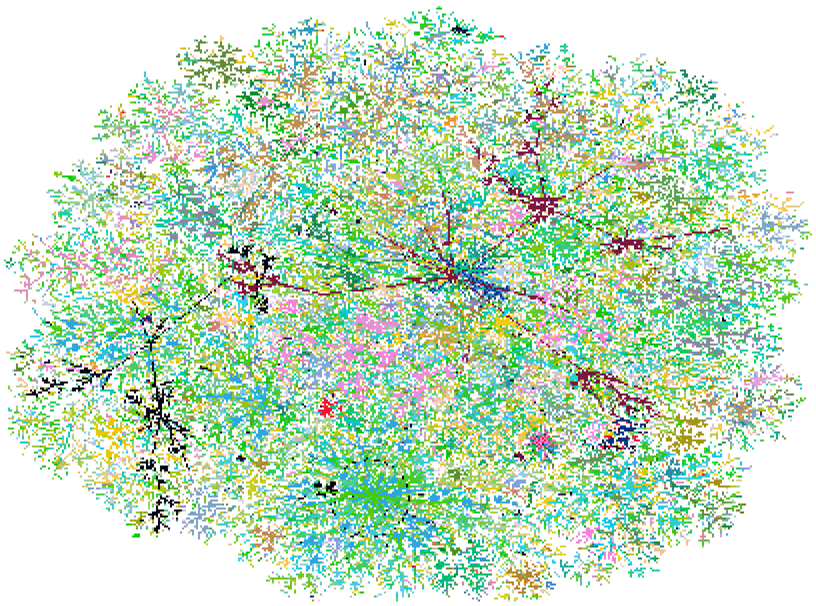
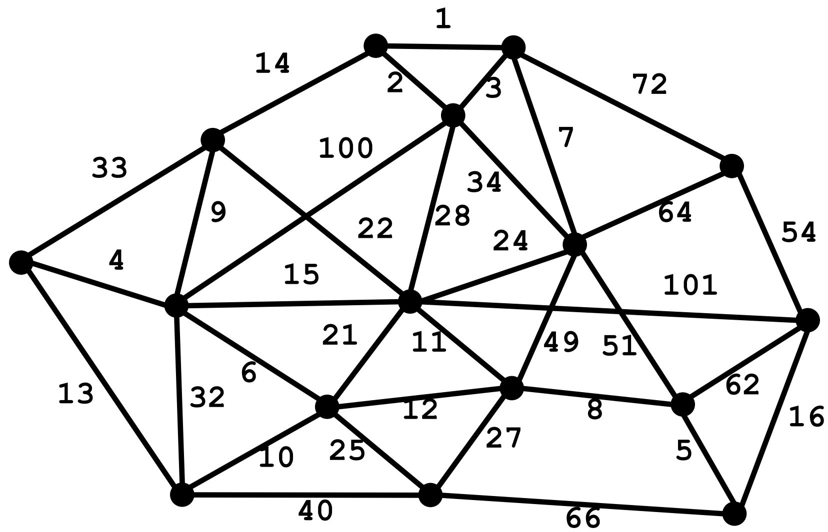


Lecture 20: Other Algorithms on Graphs

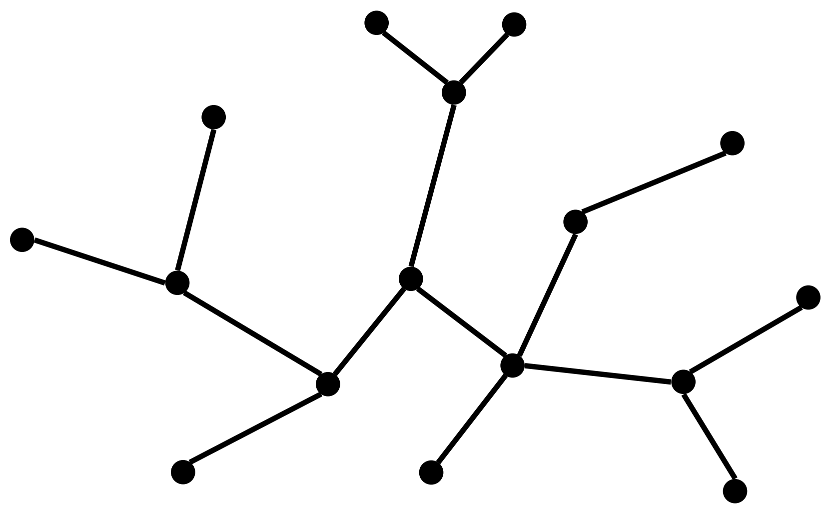
Minimum Spanning Trees

- Example Problem:
  - Nodes = neighborhoods
  - Edges = possible cable routes
  - Goal: Find lowest cost network that connects all neighborhoods
- Analogously:
  - Router network
  - Clustering
  - Component in many approximation algorithms



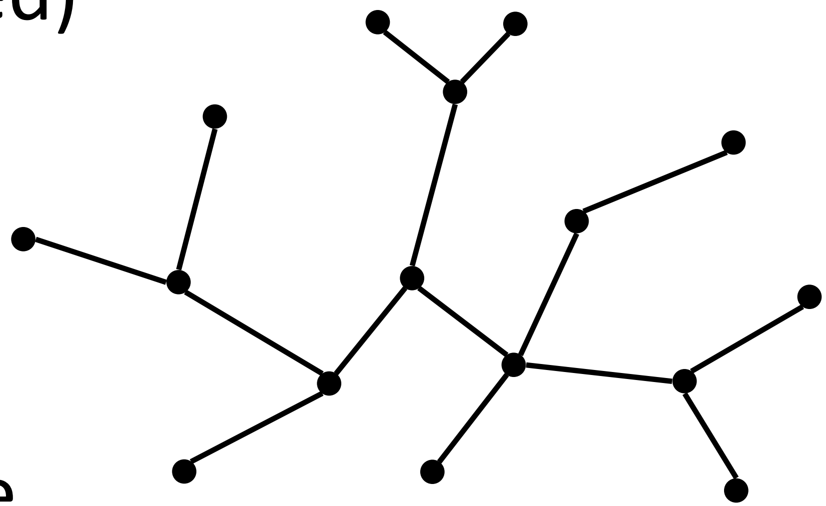
Undirected Trees

- An undirected graph is a *tree* if there is exactly one (simple) path between any pair of vertices



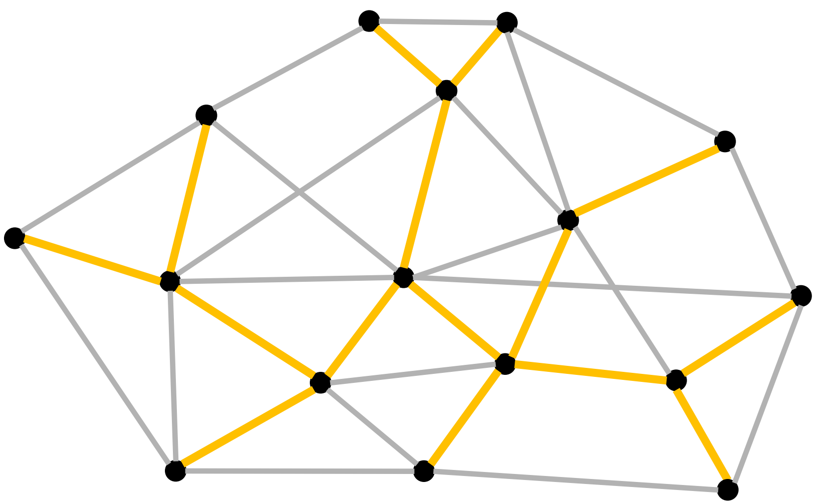
Facts About Trees

- Properties of (undirected) trees
  - $|E| = |V| - 1$
  - Connected
  - no cycles
- In fact, any two of these properties imply the third, and imply that the graph is a tree



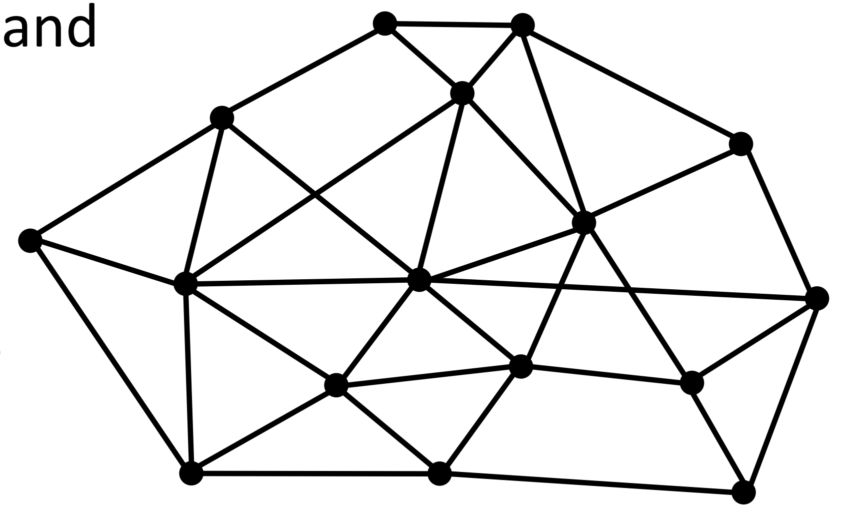
Spanning Trees

- A spanning tree of a connected undirected graph (V,E) is a subgraph (V,E') that is a tree
  - Same set of vertices V
  - $E' \subseteq E$
  - (V,E') is a tree



Finding a Spanning Tree

- A subtractive method
  - Start with the whole graph – it is connected
  - Find a cycle (how?), pick an edge on the cycle and throw it out
    → the graph is still connected (why?)
  - Repeat until no more cycles



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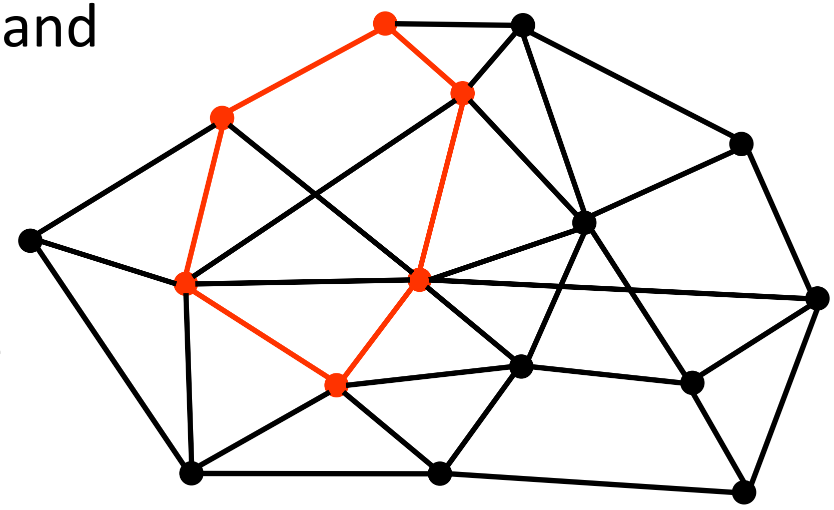

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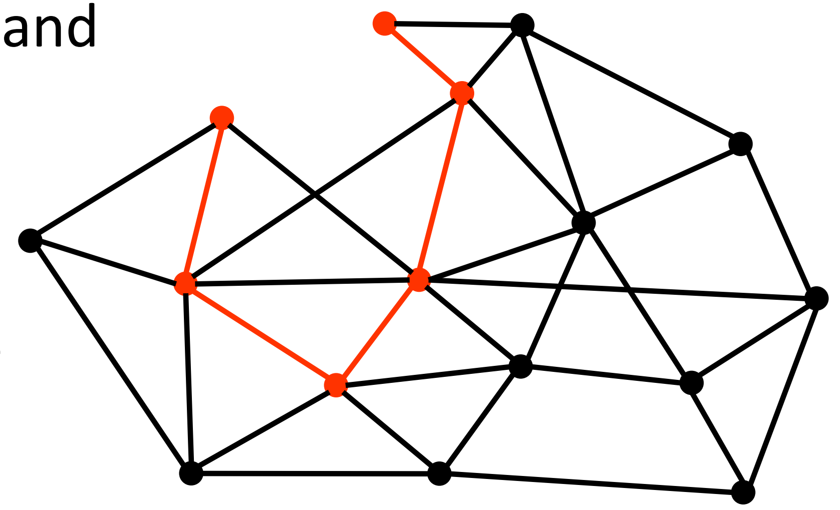
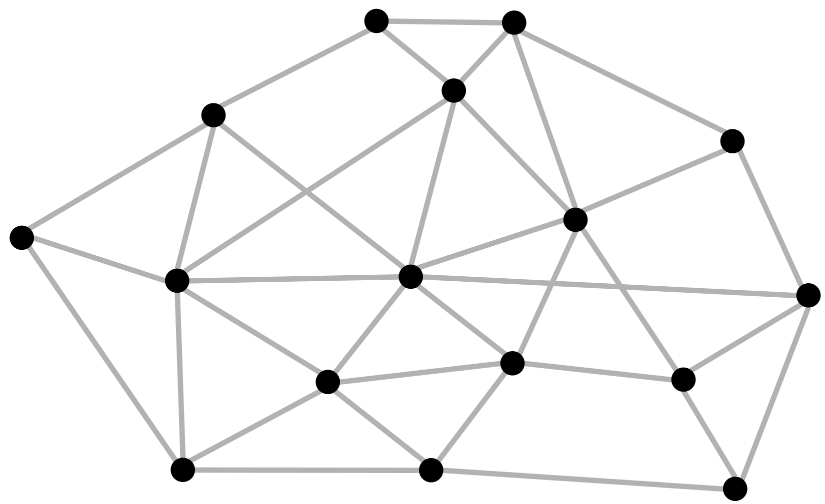


Finding a Spanning Tree

- An additive method
  - Start with no edges – there are no cycles
  - Find connected components (how?).
  - If more than one connected component, insert an edge between them
    →still no cycles (why?)
  - Repeat until only one component



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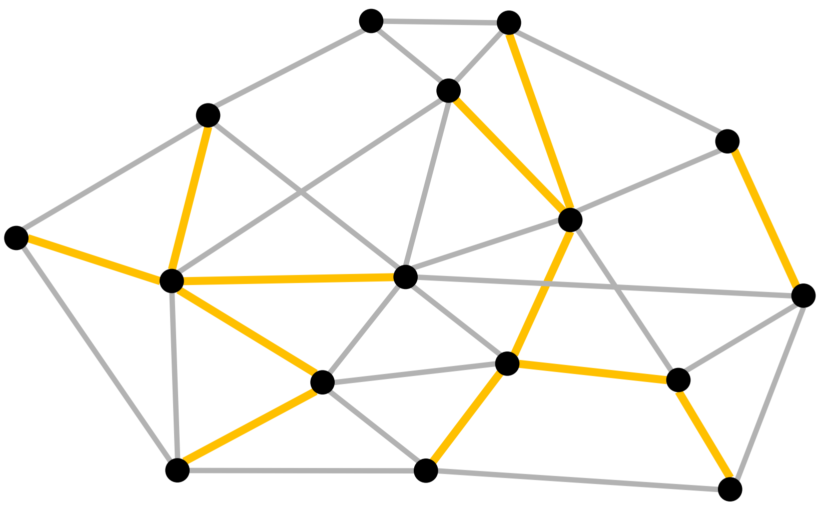

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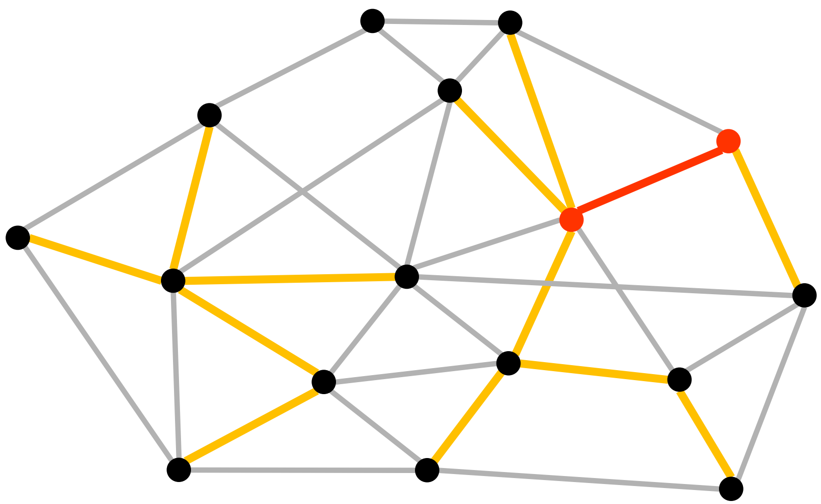

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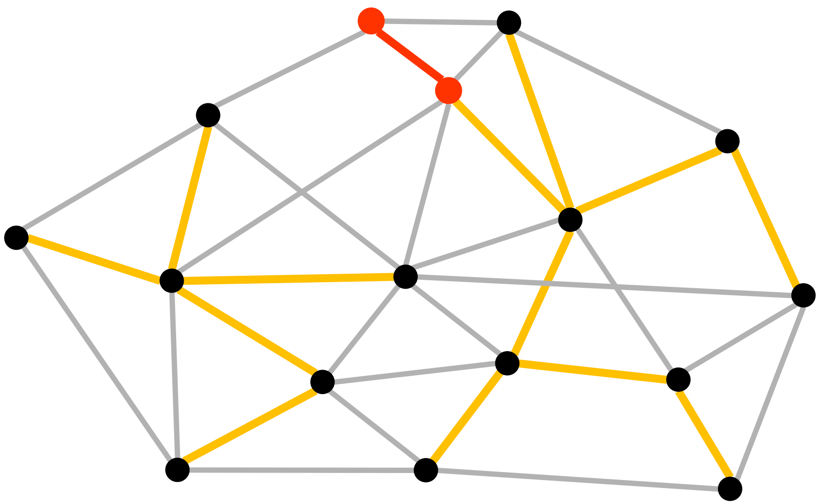

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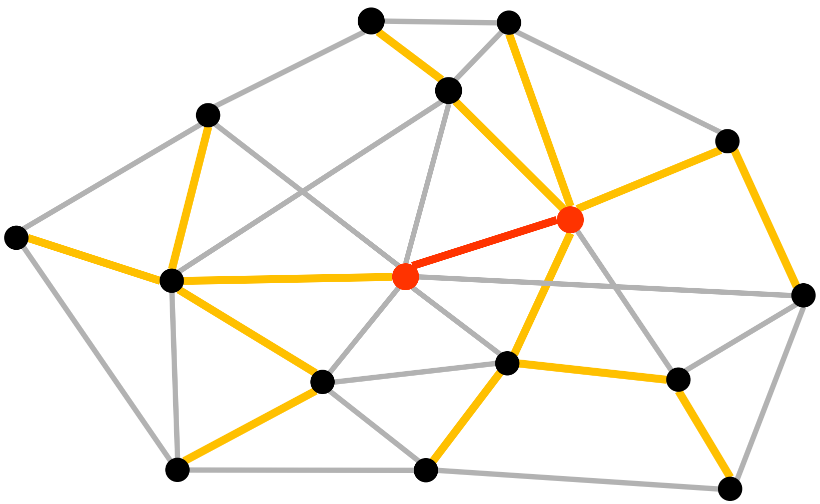

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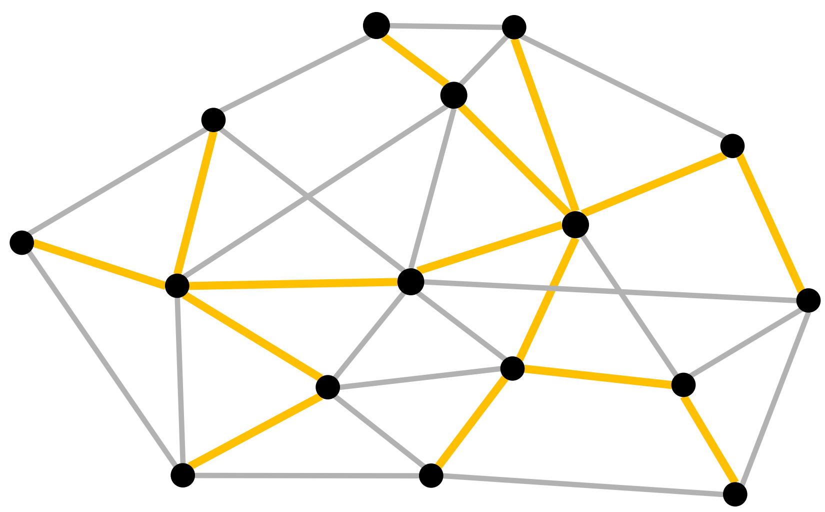
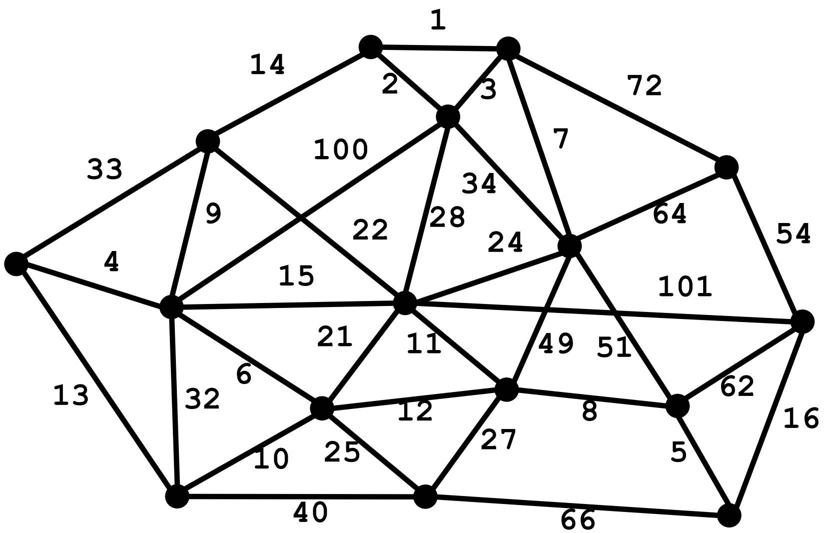


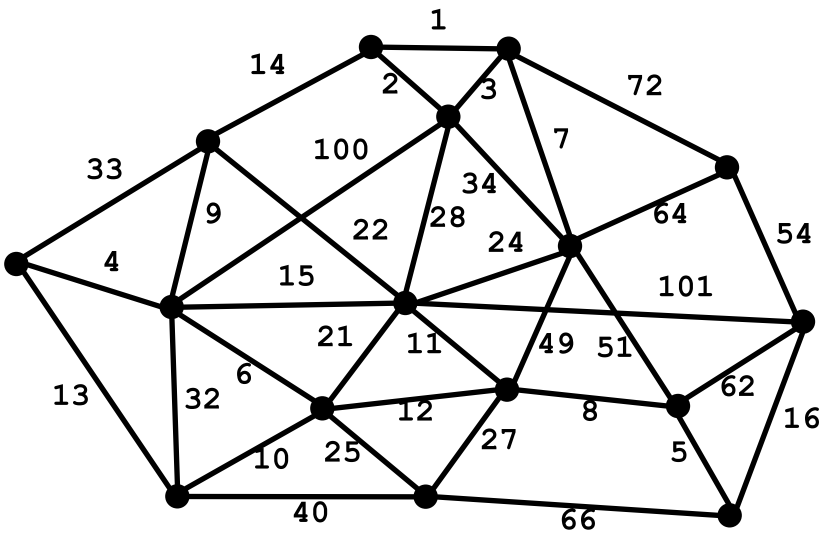
Minimum Spanning Trees

- Suppose edges are weighted, and we want a spanning tree of minimum cost (sum of edge weights)



3 Greedy Algorithms

- Algorithm A: Find a max weight edge – if it is on a cycle, throw it out, otherwise keep it



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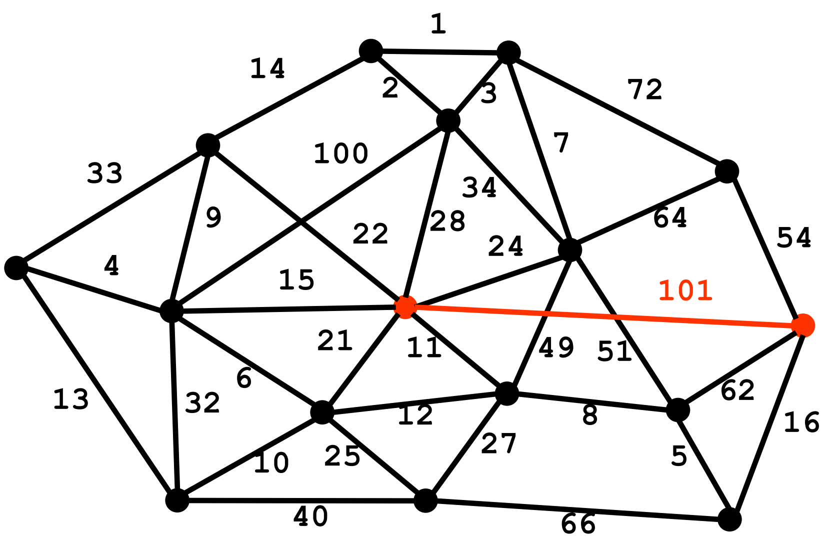

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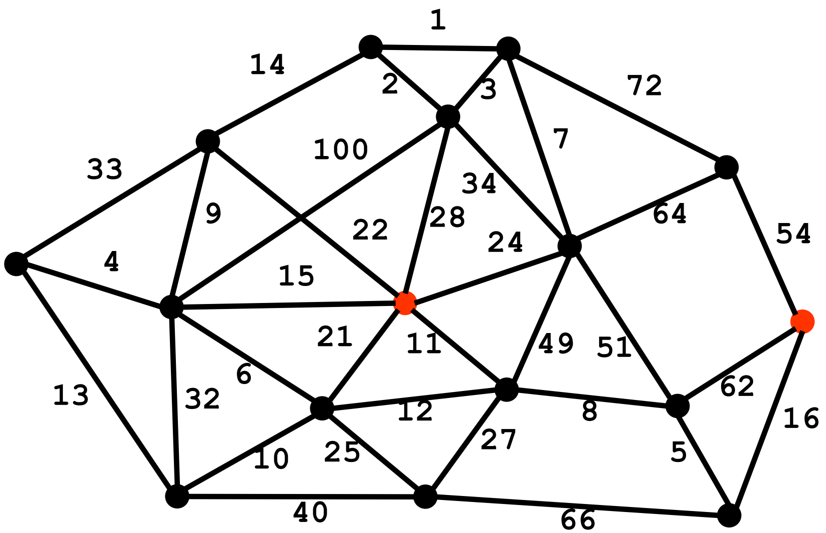

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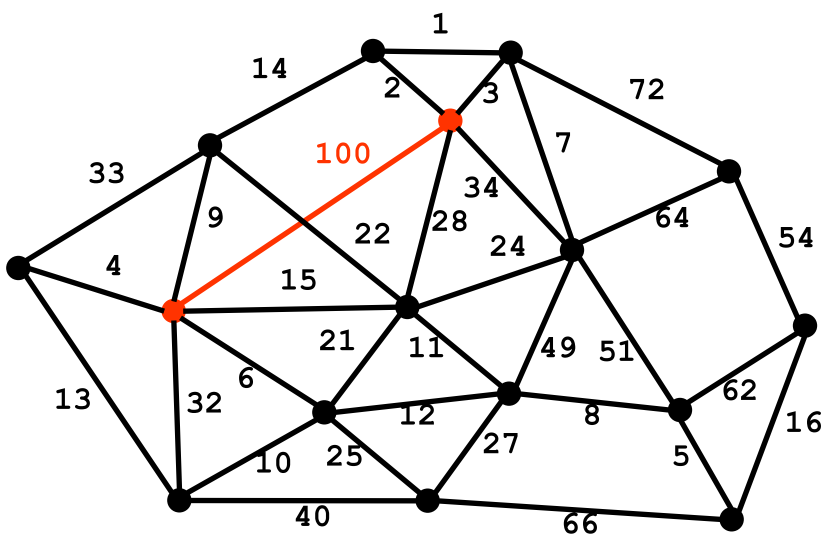

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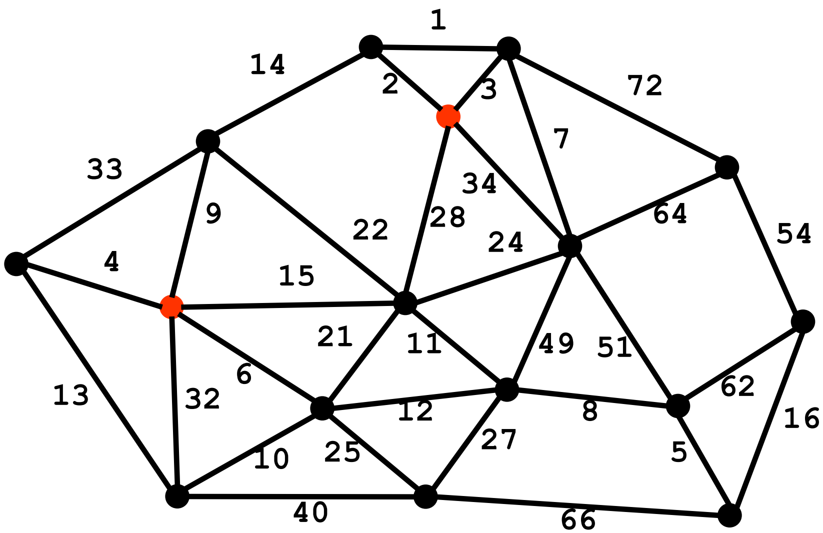

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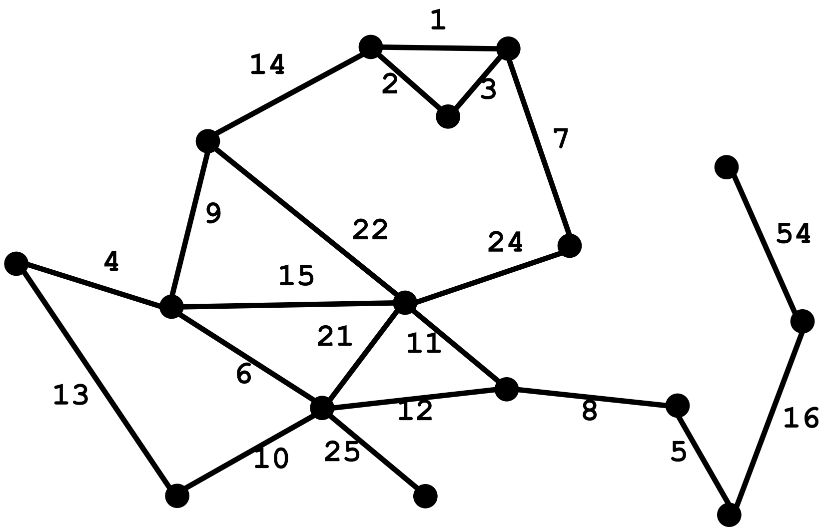

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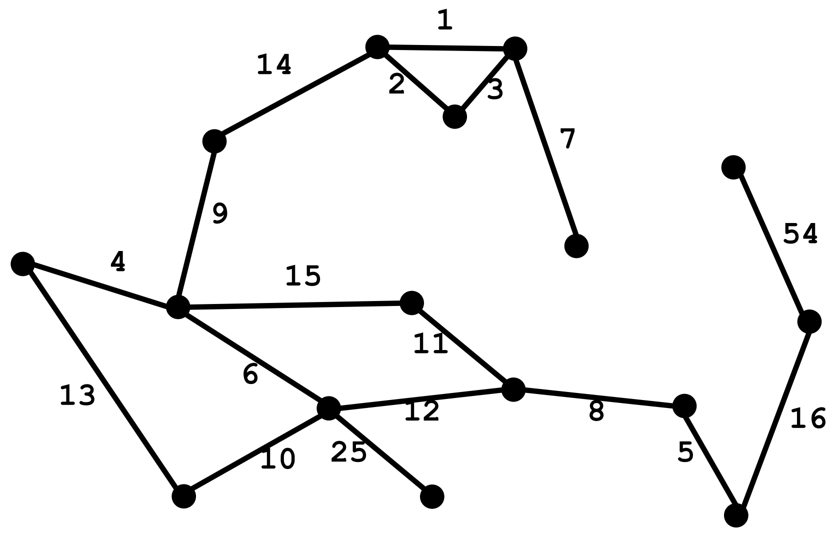

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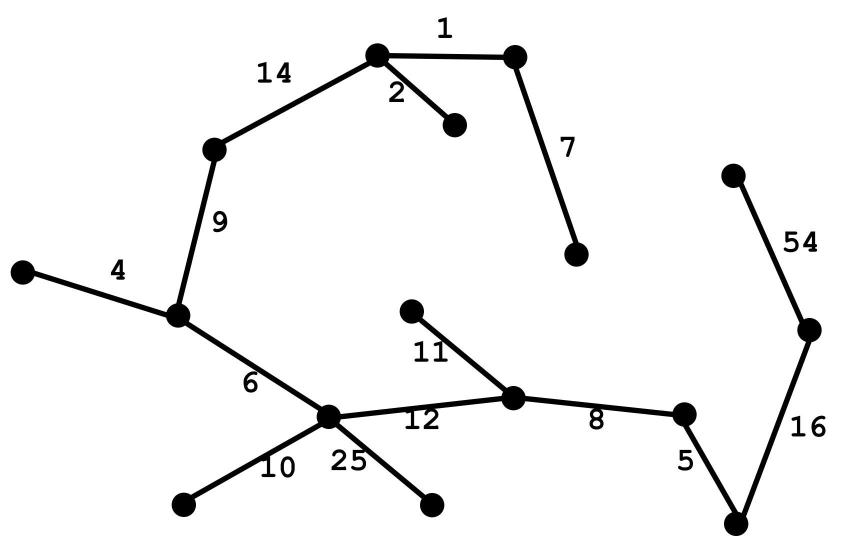
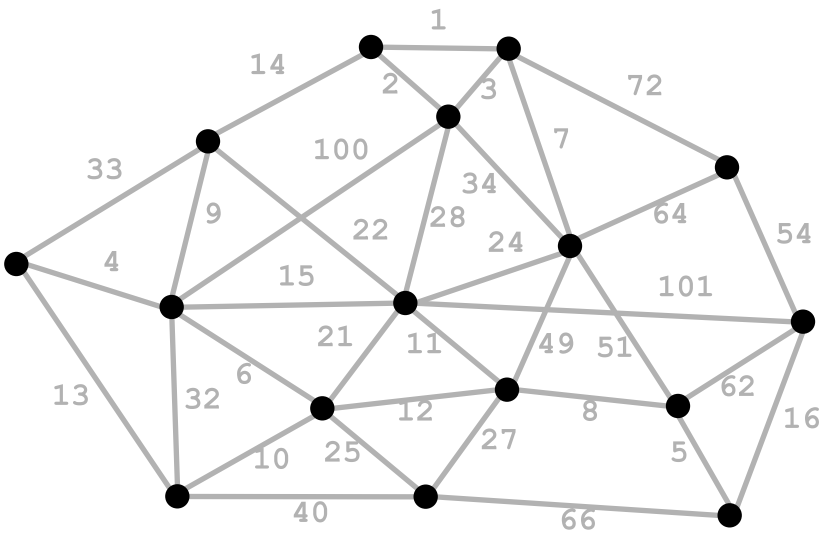


3 Greedy Algorithms

- Algorithm B: Find a min weight edge – if it forms a cycle with edges already taken, throw it out, otherwise keep it

Kruskal's algorithm



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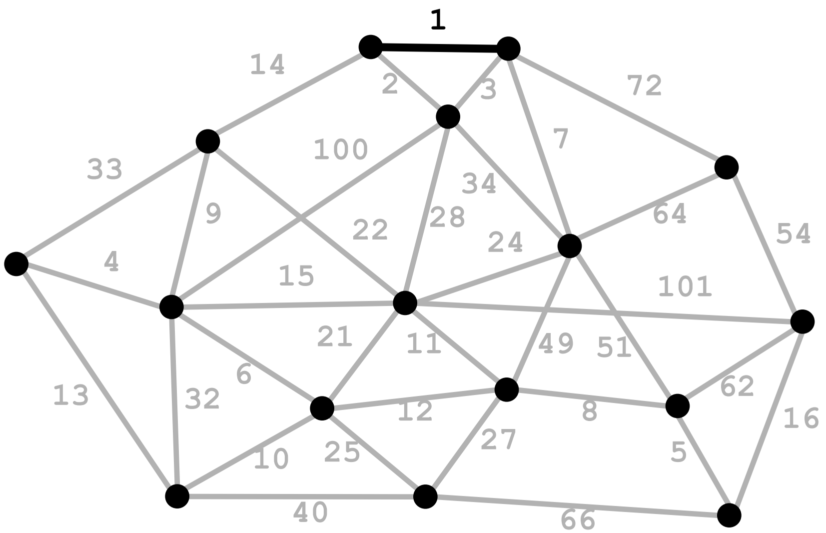

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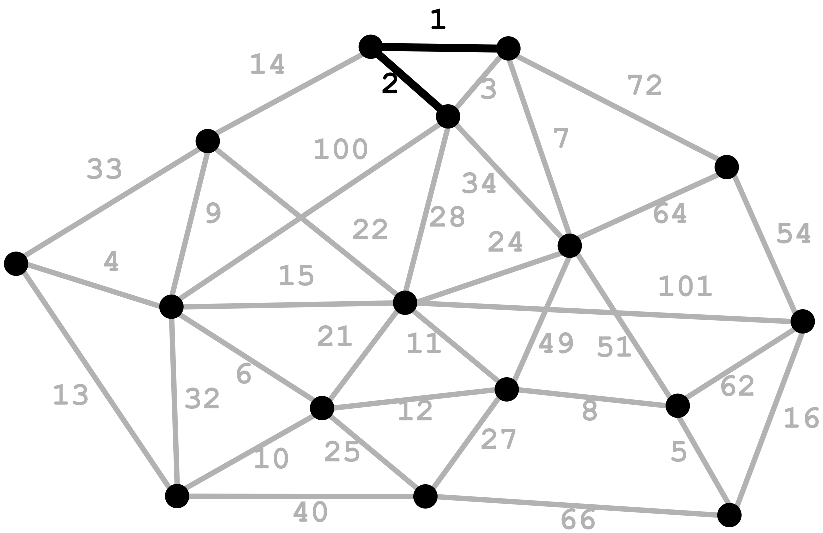

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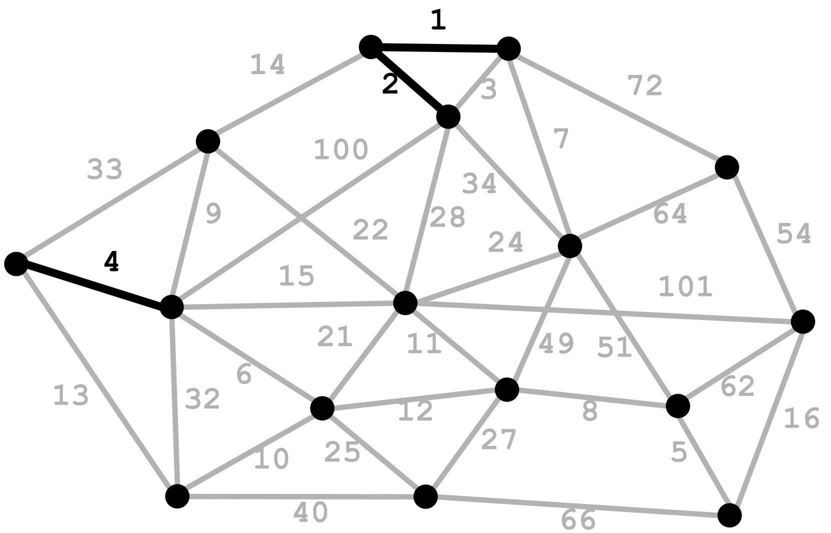

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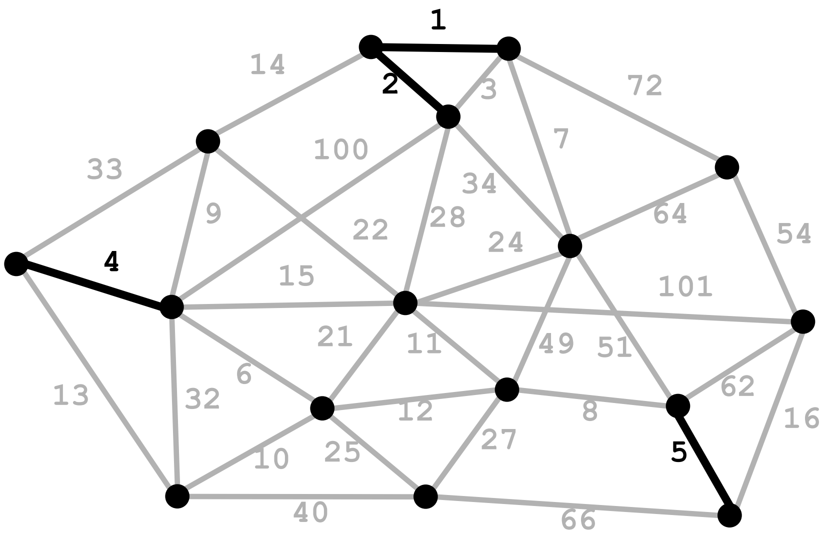

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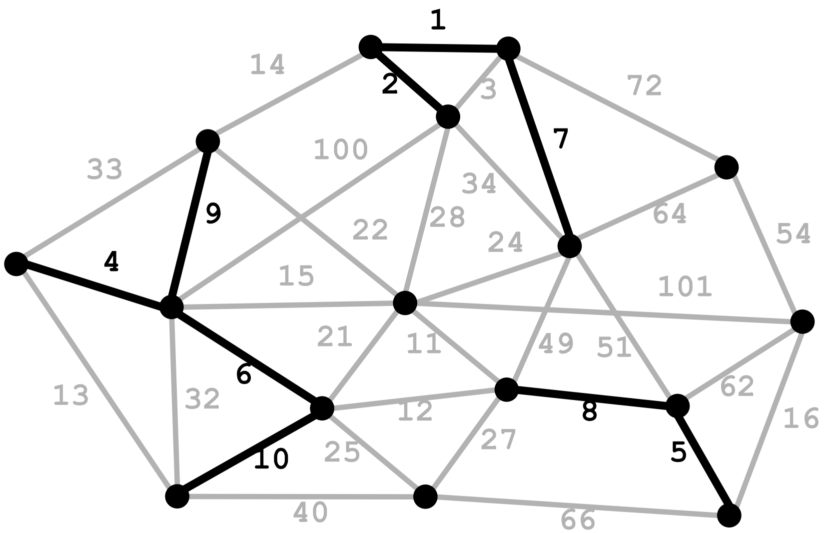

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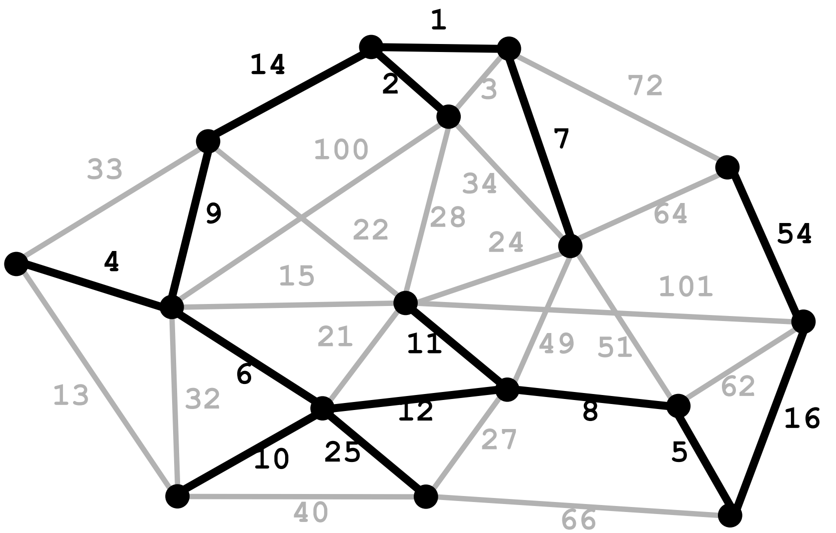
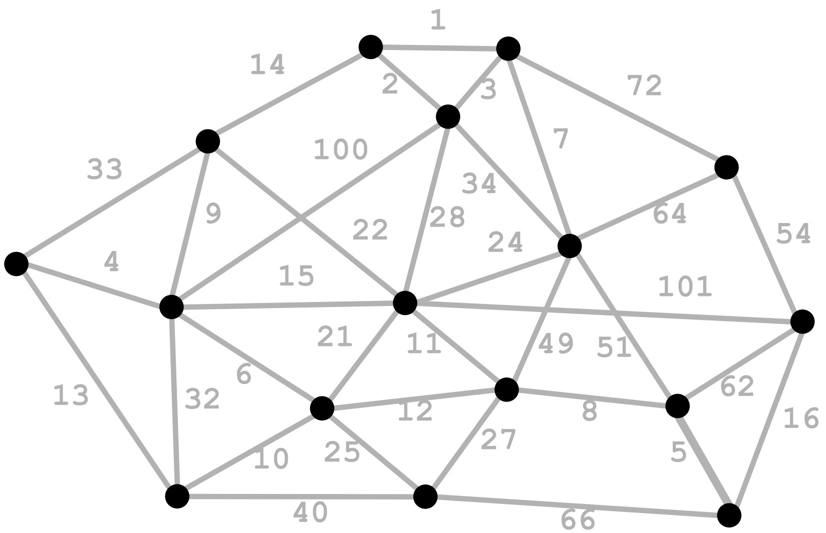


3 Greedy Algorithms

- Algorithm C: Start with any vertex, add min weight edge extending that connected component that does not form a cycle

Prim's algorithm



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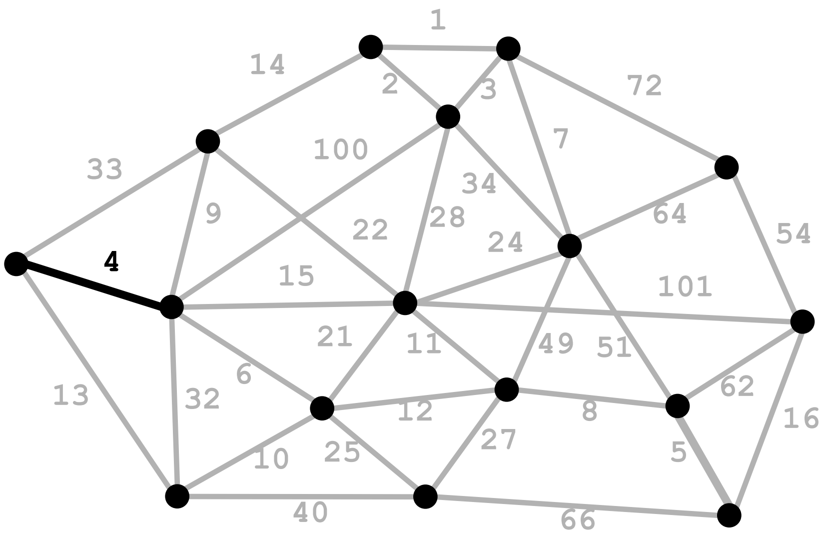

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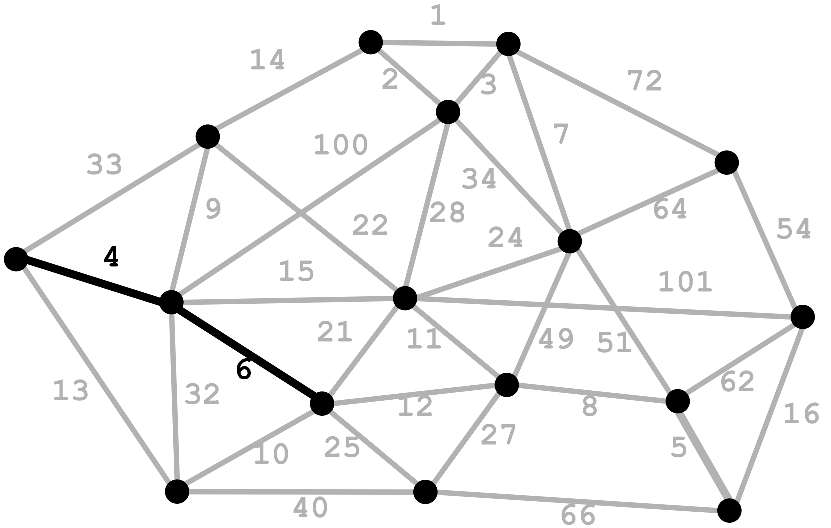

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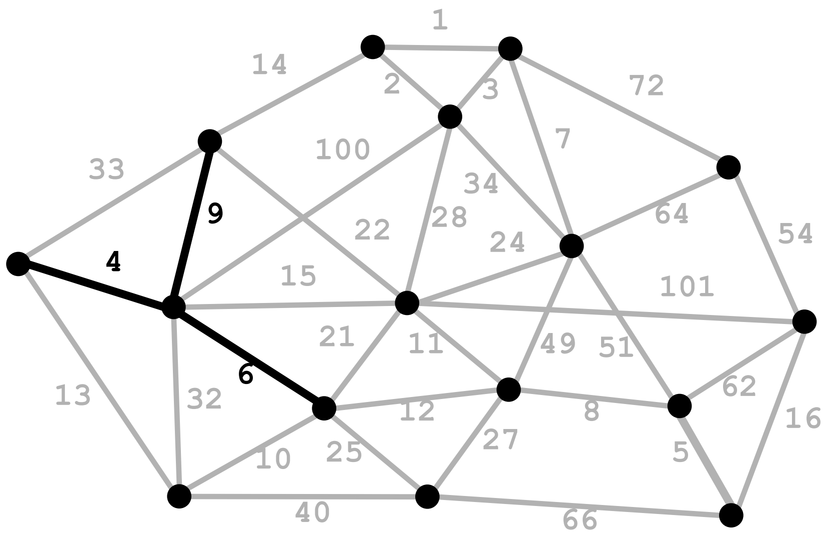

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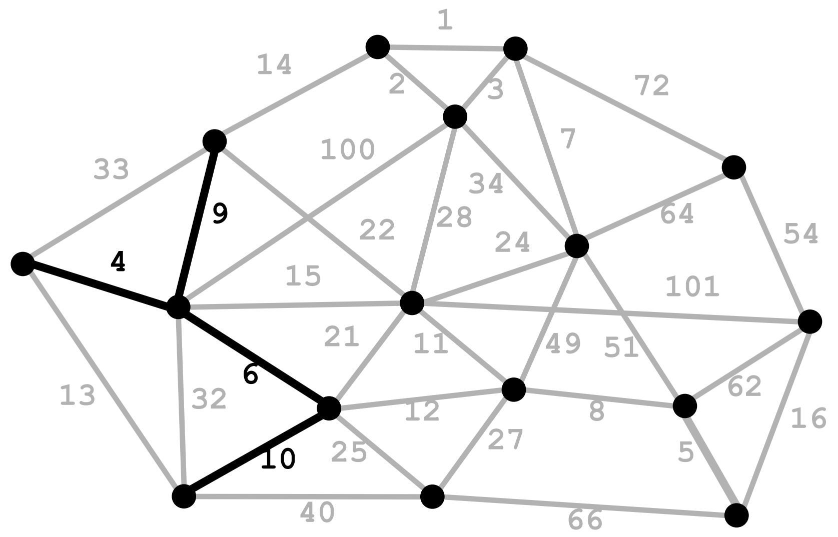

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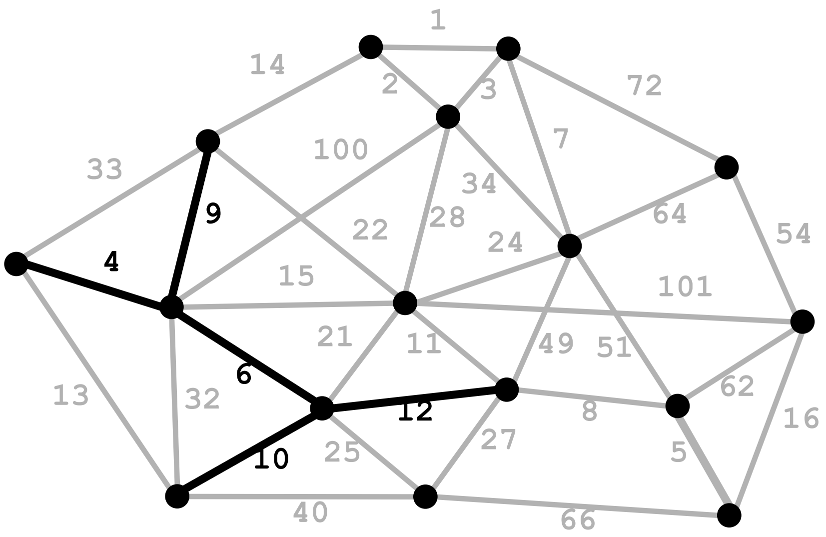

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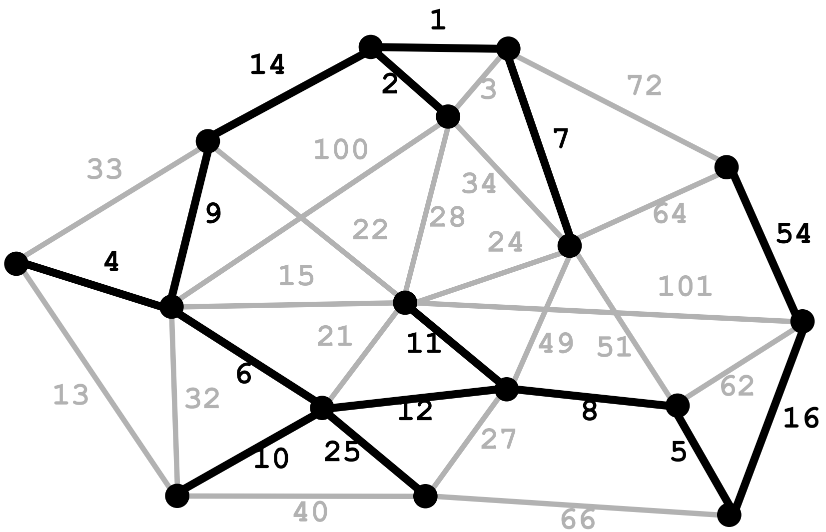
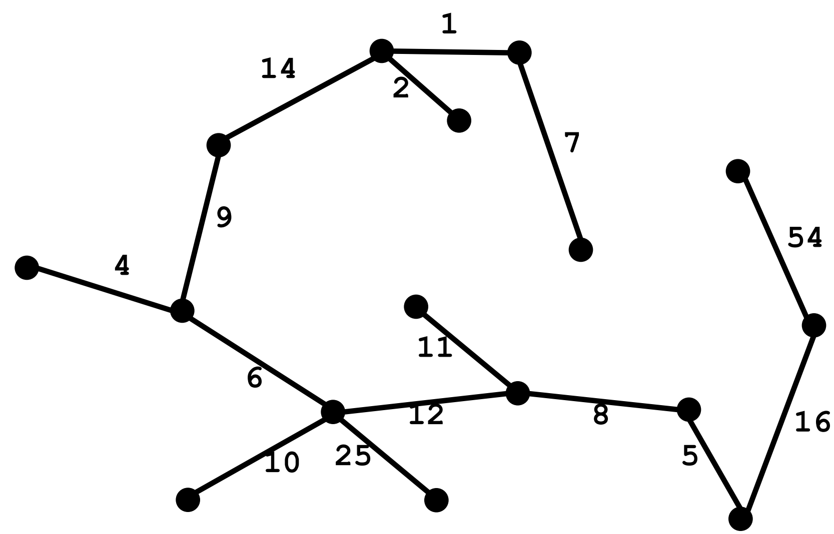


3 Greedy Algorithms

- All 3 greedy algorithms give the same minimum spanning tree (assuming distinct edge weights)



Prim's Algorithm

```
prim(s) {
   D[t] = infty for all vertices t
   D[s] = 0; //s is start vertex
   while (some vertices are unmarked) {
      v = unmarked vertex with smallest D;
      mark v;
      for (each w adj to v) {
         D[w] = min(D[w], c(v,w));
      }
   }
}
```

Min "distance" to connected component

- $O(n^2)$ for adj matrix
  - While-loop is executed n times
  - For-loop takes O(n) time
- O(m + n log n) for adj list
  - Use a PQ
  - Regular PQ produces time O(n + m log m)
  - Can improve to O(m + n log n) using a fancier heap
  - Still $O(n^2)$ if graph is not sparse

Greedy Algorithms

- These are examples of Greedy Algorithms
- The Greedy Strategy is an algorithm design technique
  - Like Divide & Conquer
- Greedy algorithms are used to solve optimization problems
  - The goal is to find the best solution
- Works when the problem has the greedy-choice property
  - A global optimum can be reached by making locally optimum choices

- Example “Change Making”:
  - Given an amount of money, find the smallest number of coins to make that amount
- Solution: Greedy Algorithm
  - Give as many large coins as you can
  - This greedy strategy produces the optimum number of coins for the US coin system
- Different money system ⇒ greedy strategy may fail

Similar Code Structures

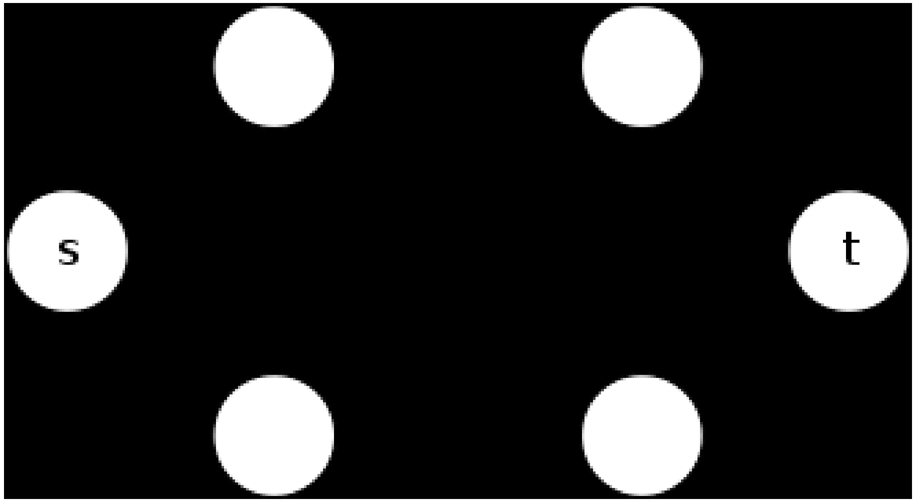
```
while (some vertices are
        unmarked) {
   v = best of unmarked
       vertices;
   mark v;
   for (each w adj to v)
       update w;
}
```

- BFS (unweighted)
  - best: next in queue
  - update: D[w] = D[v]+1
- BFS (weighted) → Dijkstra
  - best: next in PQ
  - update: D[w] = min{ D[w], D[v]+c(v,w) }
- Prim
  - best: next in PQ
  - update: D[w] = min{ D[w], c(v,w) }

Other Graph Problems

Network Flow

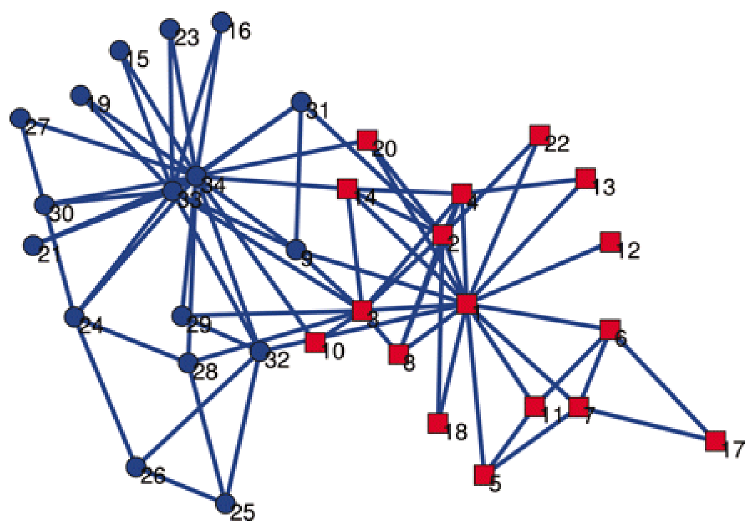
- How many "units" can flow from s to t?
  - Flow in water network
  - Traffic flow



→ Ford-Fulkerson Algorithm

Minimum Cut

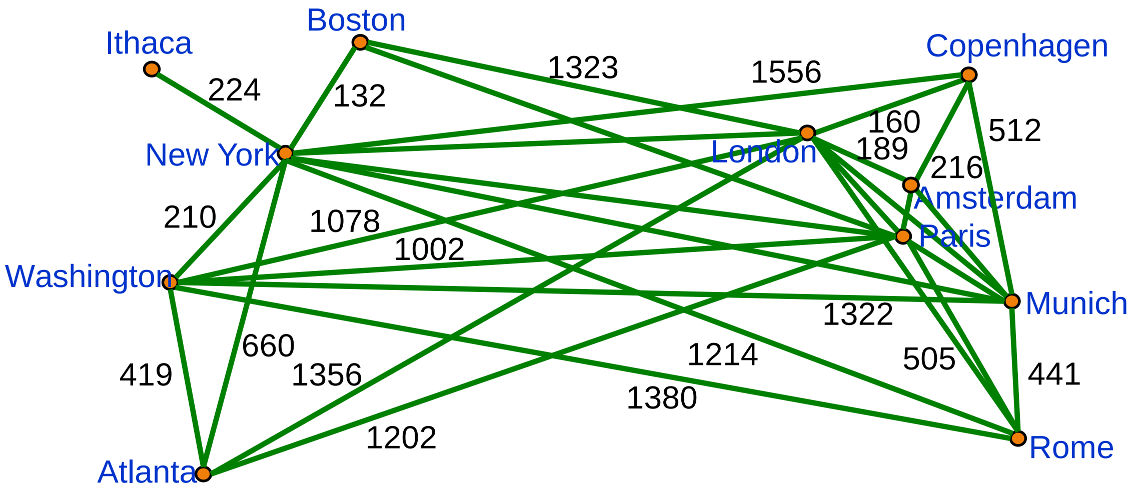
- Cut graph so that Source and Sink are separated, and the sum of the edges that are cut is minimized.
  - Traffic bottlenecks
  - Clustering in social networks



→ Duality with Maximum Flow

Traveling Salesperson

- Find a path of minimum distance that visits every city.
  - Planning and logistics
  - Microchip design



  - NP-Hard → there is probably no $O(n^k)$ algorithms