CS/ENGRD 2110
Object-Oriented Programming and Data Structures

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Lecture 10: Asymptotic Complexity and $\mathcal{O}$

What Makes a Good Algorithm?

- Suppose you have two possible algorithms or data structures that basically do the same thing; which is better?
- Well… what do we mean by better?
  - Faster?
  - Less space?
  - Easier to code?
  - Easier to maintain?
  - Required for homework?
- How do we measure time and space for an algorithm?

Sample Problem: Searching

- Determine if a *sorted* array of integers contains a given integer
- First solution: Linear Search (check each element)

```
static boolean find(int[] a, int item) {
    for (int i = 0; i < a.length; i++) {
        if (a[i] == item) return true;
    }
    return false;
}
```

```
static boolean find(int[] a, int item) {
    for (int x : a) {
        if (x == item) return true;
    }
    return false;
}
```

Sample Problem: Searching

Second solution:
Binary Search

```
static boolean find (int[] a, int item) {
    int low = 0;
    int high = a.length - 1;
    while (low <= high) {
      int mid = (low + high)/2;
      if (a[mid] < item)
         low = mid + 1;
      else if (a[mid] > item)
         high = mid - 1;
      else return true;
   }
   return false;
}
```

Linear Search vs Binary Search

- Which one is better?
  - Linear Search is easier to program
  - But Binary Search is faster… isn't it?
- How do we measure to show that one is faster than the other
  - Experiment
  - Proof
- Which inputs do we use?

- Simplifying assumption #1:
  - Use the *size* of the input rather than the input itself
  - For our sample search problem, the input size is n+1 where n is the array size
- Simplifying assumption #2:
  - Count the number of "*basic steps*" rather than computing exact times

One Basic Step = One Time Unit

- Basic step:
  - input or output of a scalar value
  - accessing the value of a scalar variable, array element, or field of an object
  - assignment to a variable, array element, or field of an object
  - a single arithmetic or logical operation
  - method invocation (not counting argument evaluation and execution of the method body)
- For a conditional, count number of basic steps on the branch that is executed
- For a loop, count number of basic steps in loop body times the number of iterations
- For a method, count number of basic steps in method body (including steps needed to prepare stack-frame)

Runtime vs Number of Basic Steps

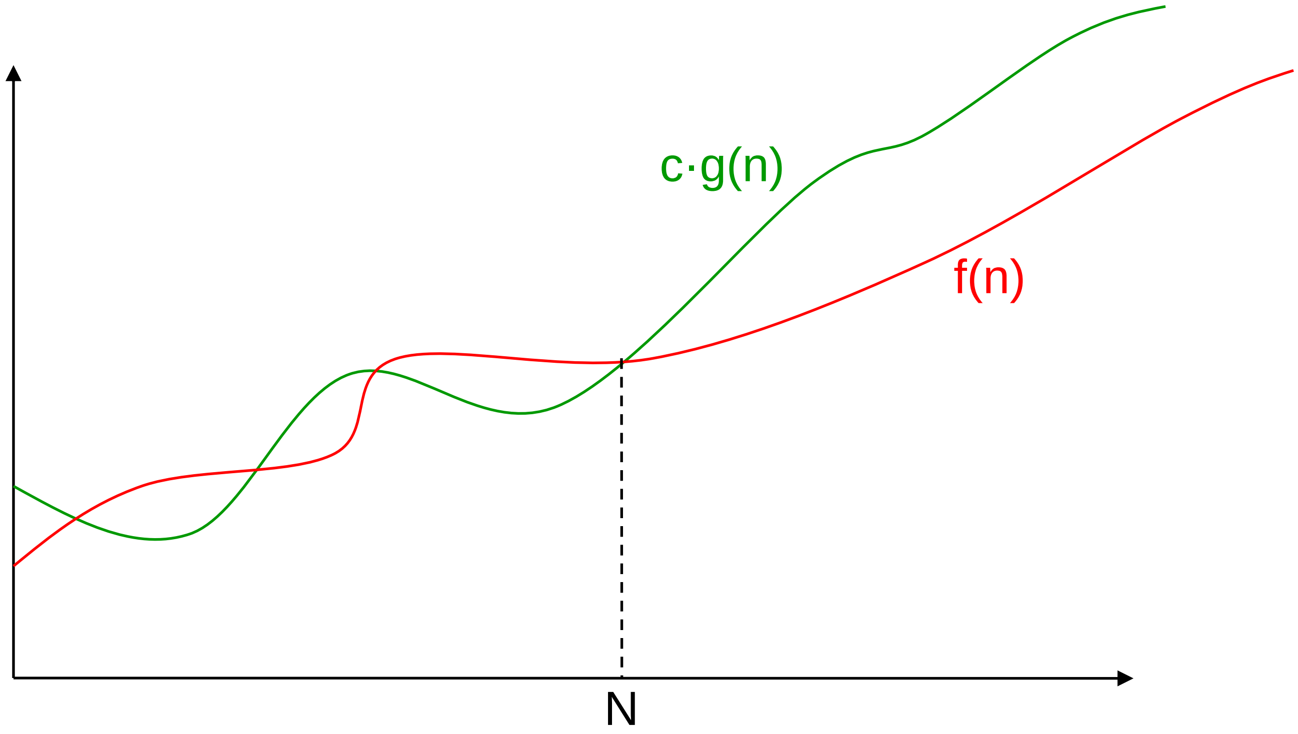
- But is this cheating?
  - The runtime is not the same as the number of basic steps
  - Time per basic step varies depending on computer, on compiler, on details of code…
- Well…yes, in a way
  - But the number of basic steps is proportional to the actual runtime
- Which is better?
  - n or n^2 time?
  - 100 n or n^2 time?
  - 10,000 n or n^2 time?
- As n gets large, multiplicative constants become less important
- Simplifying assumption #3:
  - Ignore multiplicative constants

Using Big-O to Hide Constants

- We say f(n) is order of g(n) if f(n) is bounded by a constant times g(n)
- Notation: f(n) is O(g(n))
- Roughly, f(n) is O(g(n)) means that f(n) grows like g(n) or slower, to within a constant factor
- "Constant" means fixed and independent of n

Formal definition:
f(n) is O(g(n)) if there exist constants c and N such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

A Graphical View



- To prove that f(n) is O(g(n)):
  - Find an N and c such that $f(n) \leq c\, g(n)$ for all $n \geq N$
  - We call the pair (c, N) a *witness pair* for proving that f(n) is O(g(n))

Big-O Examples

- Claim: 100 n + log n is O(n)
  - We know log n ≤ n for n ≥ 1
  - So 100 n + log n ≤ 101 n for n ≥ 1
  - So by definition, 100 n + log n is O(n) for c = 101 and N = 1
- Claim: $\log_B n$ is $O(\log_A n)$
  - since $\log_B n$ is $(\log_B A)(\log_A n)$
- Question: Which grows faster, n or log n?

Big-O Examples

- Let $f(n) = 3n^2 + 6n - 7$
  - $f(n)$ is $O(n^2)$
  - $f(n)$ is $O(n^3)$
  - $f(n)$ is $O(n^4)$
  - …
- $g(n) = 4\, n \log n + 34\, n - 89$
  - $g(n)$ is $O(n \log n)$
  - $g(n)$ is $O(n^2)$
- $h(n) = 20 \cdot 2^n + 40n$
  - $h(n)$ is $O(2^n)$
- $a(n) = 34$
  - $a(n)$ is $O(1)$

→ Only the *leading* term (the term that grows most rapidly) matters

Problem-Size Examples

- Suppose we have a computing device that can execute 1000 operations per second; how large a problem can we solve?

	1 second	1 minute	1 hour
n	1000	60,000	3,600,000
n log n	140	4893	200,000
n^2	31	244	1897
$3n^2$	18	144	1096
n^3	10	39	153
2^n	9	15	21

Commonly Seen Time Bounds

$O(1)$	constant	excellent
$O(\log n)$	logarithmic	excellent
$O(n)$	linear	good
$O(n \log n)$	n log n	pretty good
$O(n^2)$	quadratic	often OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Worst-Case/Expected-Case Bounds

- We can't possibly determine time bounds for all possible inputs of size n
- Simplifying assumption #4:
  Determine number of steps for either
  - worst-case: Determine how much time is needed for the worst possible input of size n
  - expected-case: Determine how much time is needed on average for all inputs of size n

Our Simplifying Assumptions

- Use the size of the input rather than the input itself – n
- Count the number of "basic steps" rather than computing exact times
- Multiplicative constants and small inputs ignored (order-of, big-O)
- Determine number of steps for either
  - worst-case
  - expected-case

→ These assumptions allow us to analyze algorithms effectively and easily

Worst-Case Analysis of Searching

- Linear Search

```
static boolean find (int[] a, int item)
    {
   for (int i = 0; i < a.length; i++) {
      if (a[i] == item) return true;
   }
   return false;
}
```

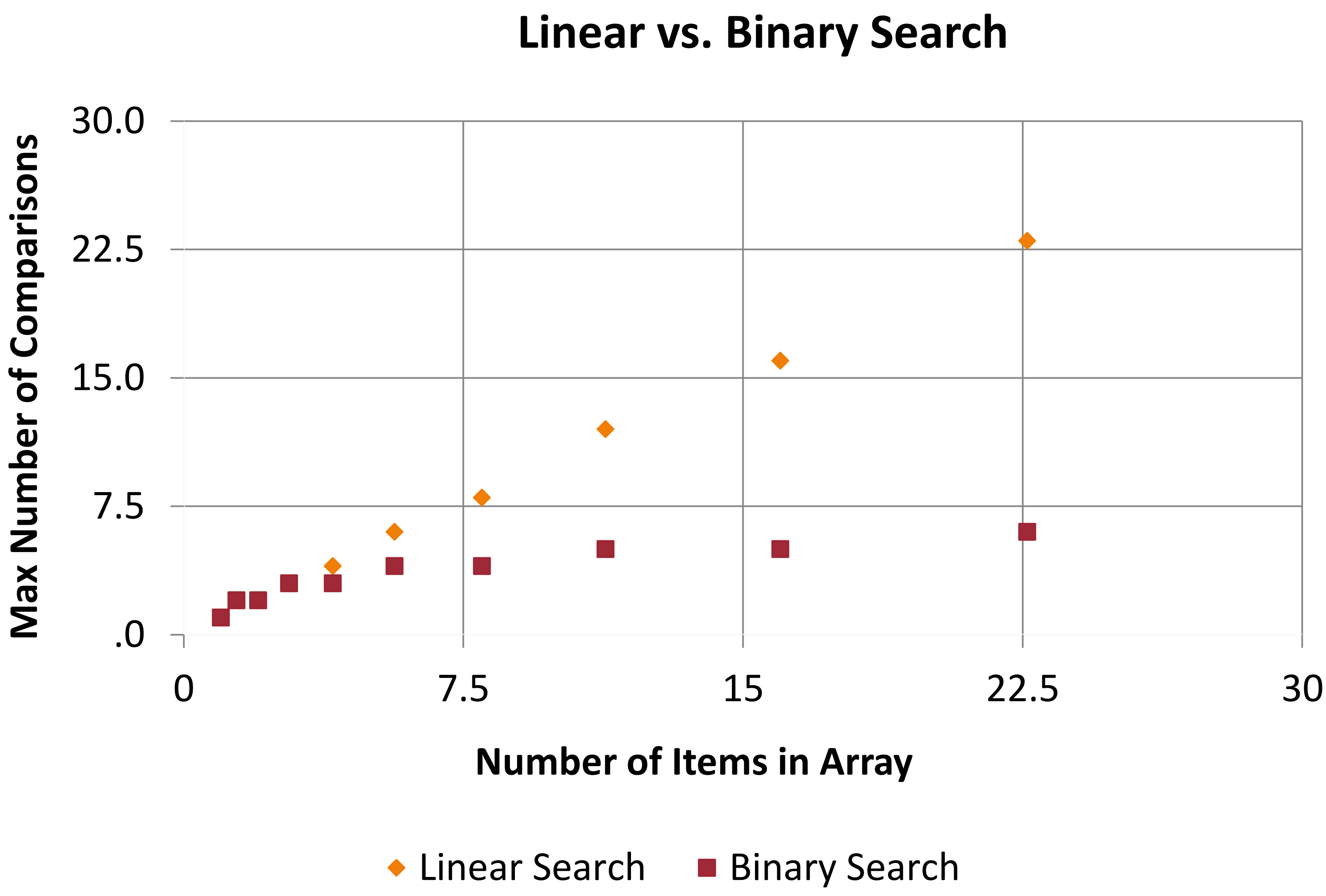
worst-case time = O(n)

- Binary Search

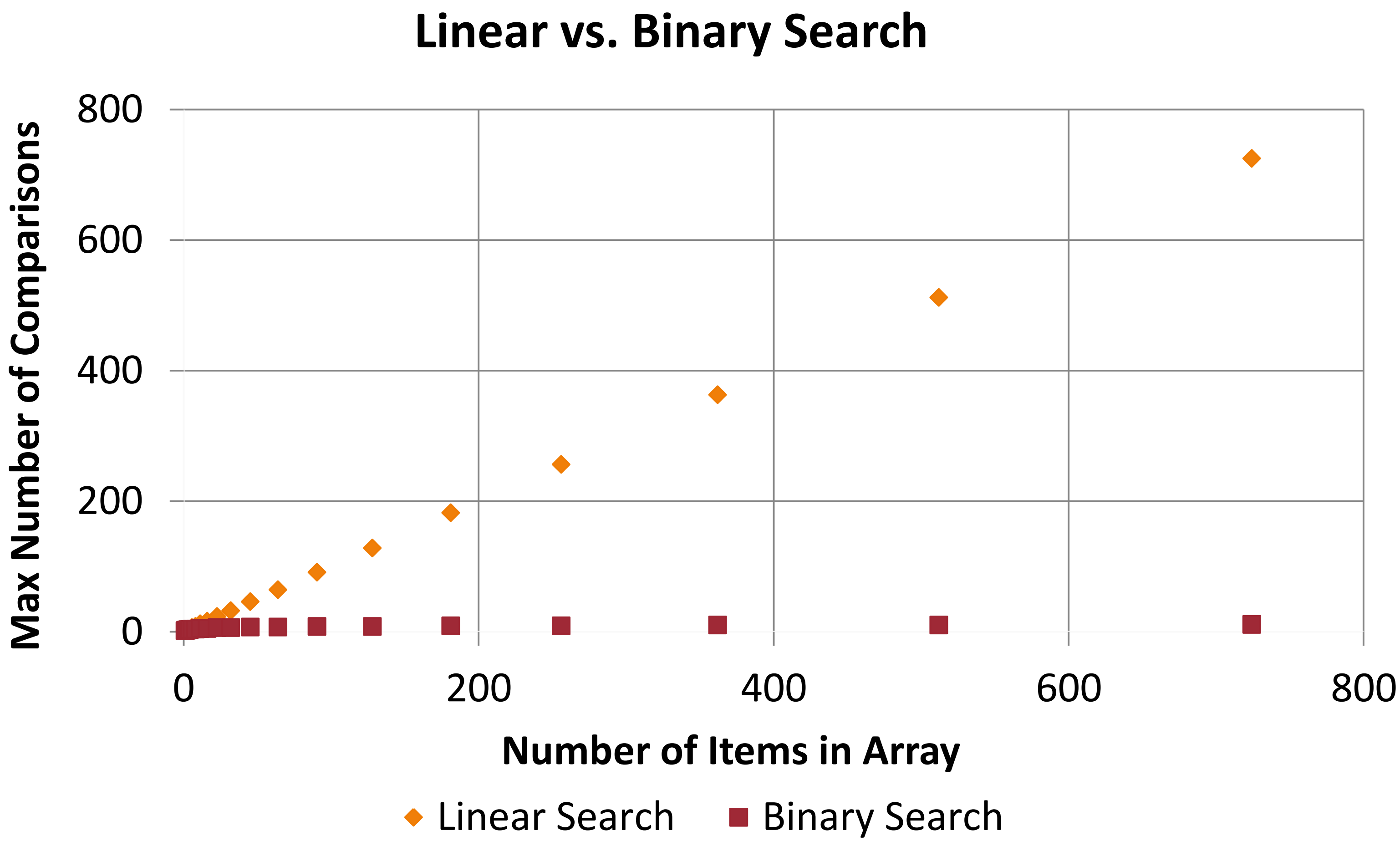
```
static boolean find (int[] a, int item) {
   int low = 0;
   int high = a.length - 1;
   while (low <= high) {
      int mid = (low + high)/2;
      if (a[mid] < item)
         low = mid+1;
      else if (a[mid] > item)
         high = mid - 1;
      else return true;
   }
   return false;
}
```

worst-case time = O(log n)

Comparison of Algorithms



Comparison of Algorithms



Analysis of Matrix Multiplication

- Code for multiplying n-by-n matrices A and B:
  - By convention, matrix problems are measured in terms of n, the number of rows and columns
    - Note that the input size is really $2n^2$, not n
  - Worst-case time is $O(n^3)$
  - Expected-case time is also $O(n^3)$

```
for (i = 0; i < n; i++)
        for (j = 0; j < n; j++) {
                C[i][j] = 0;
                for (k = 0; k < n; k++)
                        C[i][j] += A[i][k]*B[k][j];
        }
```

Remarks

- Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity
  - For example, you can usually ignore everything that is not in the innermost loop. Why?

- Main difficulty:
  - Determining runtime for recursive programs

Why Bother with Runtime Analysis?

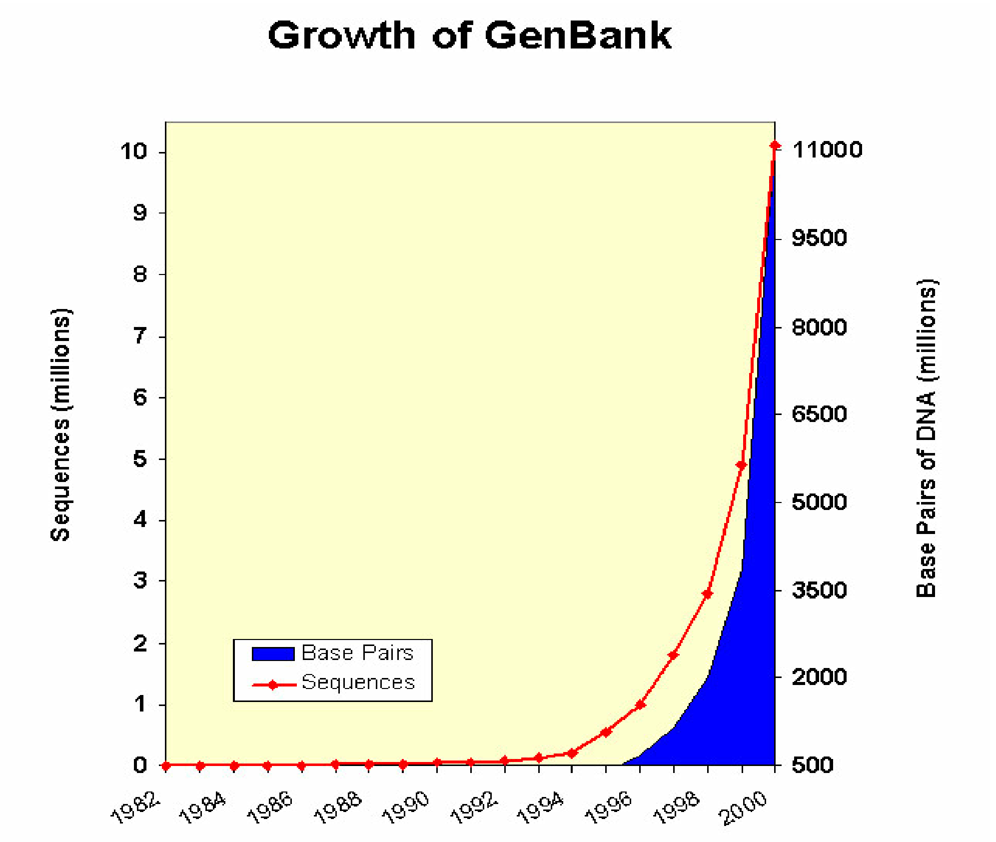
- Computers are so fast these days that we can do whatever we want using just simple algorithms and data structures, right?
- Well…not really – data-structure/algorithm improvements can be a very big win

- Scenario:
  - A runs in n^2 msec
  - A' runs in $n^2/10$ msec
  - B runs in 10 n log n msec
- Problem of size $n=10^3$
  - A: 10^3 sec ≈ 17 minutes
  - A': 10^2 sec ≈ 1.7 minutes
  - B: 10^2 sec ≈ 1.7 minutes
- Problem of size $n=10^6$
  - A: 10^9 sec ≈ 30 years
  - A': 10^8 sec ≈ 3 years
  - B: $2 \cdot 10^5$ sec ≈ 2 days
- 1 day = 86,400 sec ≈ 105 sec
- 1,000 days ≈ 3 years

Algorithms for the Human Genome

- Human genome
  = 3.5 billion nucleotides
  ~ 1 Gb

- @1 base-pair instructions/⌠ sec
  - n^2 → 388445 years
  - n log n → 30.824 hours
  - n → 1 hour



Limitations of Runtime Analysis

- Big-O can hide a very large constant
  - Example: selection
  - Example: small problems
- The specific problem you want to solve may not be the worst case
  - Example: Simplex method for linear programming
- Your program may not be run often enough to make analysis worthwhile
  - Example: one-shot vs. every day
  - You may be analyzing and improving the wrong part of the program
- Should also use profiling tools

Summary

- Asymptotic complexity
  - Used to measure of time (or space) required by an algorithm
  - Measure of the *algorithm*, not the *problem*
- Searching a sorted array
  - Linear search: O(n) worst-case time
  - Binary search: O(log n) worst-case time
- Matrix operations:
  - Note: n = number-of-rows = number-of-columns
  - Matrix-vector product: $O(n^2)$ worst-case time
  - Matrix-matrix multiplication: $O(n^3)$ worst-case time
- More later with sorting and graph algorithms