Inductive proof that $\text{fib}(n) = \frac{\varphi^n - \varphi'^n}{\sqrt{5}}$

Let $\text{fib}(n)$ be the $n$th Fibonacci number defined by the recurrence:

$$\text{fib}(0) = 0 \quad \text{fib}(1) = 1 \quad \text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2), \quad n \geq 2. \quad (1)$$

The Fibonacci sequence is 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Let $\varphi$ and $\varphi'$ be the roots of the quadratic polynomial $x^2 - x - 1$:

$$\varphi = \frac{1 + \sqrt{5}}{2} \quad \varphi' = \frac{1 - \sqrt{5}}{2}$$

The positive root $\varphi$ is called the golden ratio. Being roots of the polynomial $x^2 - x - 1$ says that

$$\varphi^2 = \varphi + 1 \quad \varphi'^2 = \varphi' + 1.$$

Multiplying both sides of these equations by $\varphi^{n-2}$ and $\varphi'^{n-2}$ respectively,

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} \quad \varphi'^n = \varphi'^{n-1} + \varphi'^{n-2}. \quad (2)$$

Now we proceed by induction on $n$.

**Basis** For $n = 0$ and $n = 1$,

$$\text{fib}(0) = 0 = \frac{1 - 1}{\sqrt{5}} = \frac{\varphi^0 - \varphi'^0}{\sqrt{5}},$$

$$\text{fib}(1) = 1 = \frac{1 + \sqrt{5} - (1 - \sqrt{5})}{2\sqrt{5}} = \frac{\varphi^1 - \varphi'^1}{\sqrt{5}}.$$

**Induction Step** For $n \geq 2$,

$$\text{fib}(n) = \text{fib}(n-1) + \text{fib}(n-2) \quad \text{by (1)}$$

$$= \frac{\varphi^{n-1} - \varphi'^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \varphi'^{n-2}}{\sqrt{5}} \quad \text{by the induction hypothesis}$$

$$= \frac{\varphi^{n-1} + \varphi^{n-2} - (\varphi'^{n-1} + \varphi'^{n-2})}{\sqrt{5}}$$

$$= \frac{\varphi^n - \varphi'^n}{\sqrt{5}} \quad \text{by (2)}. $$