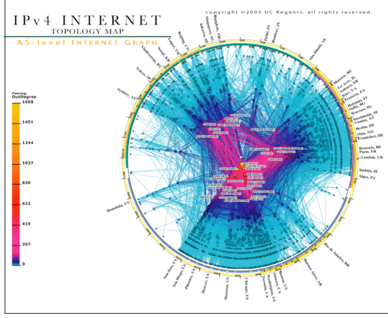
Graphs & Graph Algorithms

Lecture 22

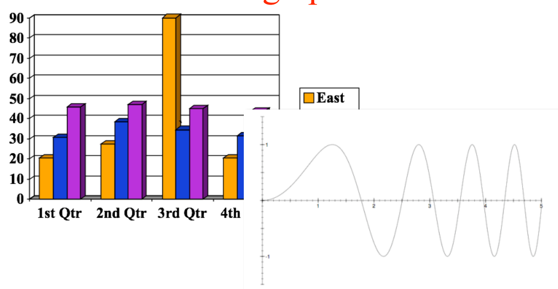
CS211 – Spring 2006



Announcements

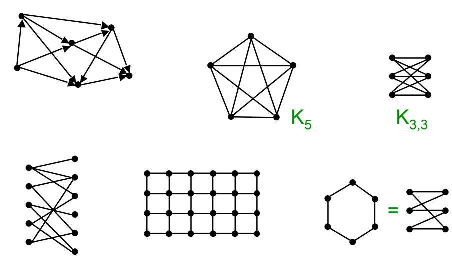
- A6 released today: Risk
  - Start designing with your partner asap…
- Prelim 2 Tuesday

Not graphs



...not the kind we mean, anyway

These are Graphs



Applications of Graphs

- Communication networks
- Routing and shortest path problems
- Commodity distribution (flow)
- Traffic control
- Resource allocation
- Geometric modeling
- ...

Graphs can be used to represent many things

⇒ graph algorithms have many applications…

Graph Definitions

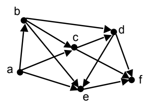
- A *directed graph* (or *digraph*) is a pair (V,E) where
  - V is a set
  - E is a set of ordered pairs (u,v) where $u,v \in V$
    - Usually require $u \neq v$ (no self-loops)

- An element of V is called a *vertex* (pl. *vertices*) or *node*
- An element of E is called an *edge* or *arc*

- $|V|$ = size of V, often denoted n
- $|E|$ = size of E, often denoted m

Example *Directed* Graph

Example:



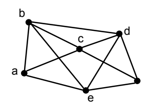
V = {a,b,c,d,e,f}
E = {(a,b), (a,c), (a,e), (b,c), (b,d), (b,e), (c,d), (c,f), (d,e), (d,f), (e,f)}

|V| = 6, |E| = 11

Example *Undirected* Graph

An *undirected graph* is just like a directed graph, except the edges are *unordered pairs* (*sets*) {u,v}

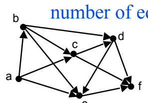
Example:

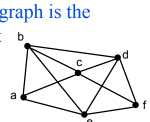


V = {a,b,c,d,e,f}
E = {{a,b}, {a,c}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,f}, {d,e}, {d,f}, {e,f}}

Some Graph Terminology

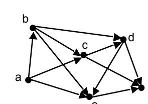
- Vertices u and v are called the *source* and *sink* of the directed edge (u,v), respectively
- Vertices u and v are called the *endpoints* of (u,v)
- Two vertices are *adjacent* if they are connected by an edge
- The *outdegree* of a vertex u in a directed graph is the number of edges for which u is the source
- The *indegree* of a vertex v in a directed graph is the number of edges for which v is the sink
- The *degree* of a vertex u in an *undirected* graph is the number of edges of which u is an endpoint



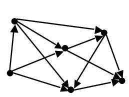


More Graph Terminology

- A *path* is a sequence $v_0,v_1,v_2,\ldots,v_p$ of vertices such that $(v_i,v_{i+1}) \in E$, $0 \le i \le p-1$
- The *length* of a path is its number of edges
  - In this example, the length is 5
- A path is *simple* if it does not repeat any vertices
- A *cycle* is a path $v_0,v_1,v_2,\ldots,v_p$ such that $v_0 = v_p$
- A cycle is *simple* if it does not repeat any vertices except the first and last
- A graph is *acyclic* if it has no cycles
- A *directed acyclic graph* is called a *dag*

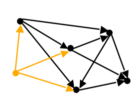


Is this a dag?



- Intuition: If it's a dag, there should be a "first" vertex (i.e., a vertex with indegree zero)
- This idea leads to an algorithm
  - A digraph is a dag if and only if we can iteratively delete indegree-0 vertices until the graph disappears

Is this a dag?



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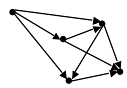

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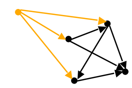

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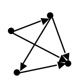

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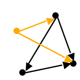

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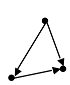

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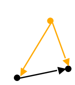

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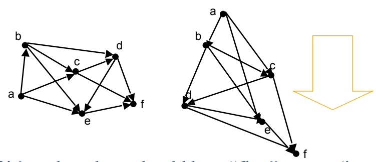

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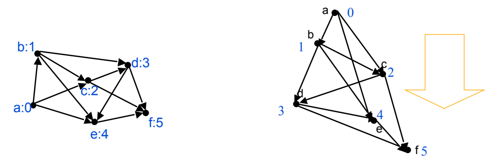
- This idea leads to an algorithm
  - A digraph is a dag if and only if we can iteratively delete indegree-0 vertices until the graph disappears
- Dag is like a tree but nodes can have multiple parents

Topological Sort

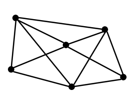
- Just computed a *topological sort* of the dag
  - A numbering of the vertices such that all edges go from lower- to higher-numbered vertices



- A dag: generalization of a tree w/ multiple parents
- Useful in job scheduling with constraints

Graph Coloring

- A *coloring* of an undirected graph is an assignment of a color to each node such that no two adjacent vertices get the same color



- How many colors are needed to color this graph?

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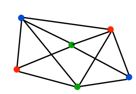
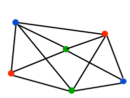


- How many colors are needed to color this graph?
  - 3

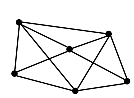
An Application of Coloring

- Vertices are jobs
- Edge (u,v) is present if jobs u and v each require access to the same shared resource, and thus cannot execute simultaneously
- Colors are time slots to schedule the jobs
- Minimum number of colors needed to color the graph = minimum number of time slots required



Planarity

- A graph is *planar* if it can be embedded in the plane with no edges crossing



- Is this graph planar?

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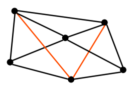

- Is this graph planar?
  - Yes

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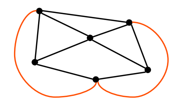
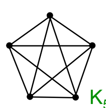


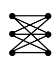
- Is this graph planar?
  - Yes

Detecting Planarity

Kuratowski's Theorem



K_5



$K_{3,3}$

A graph is planar if and only if it does not contain a copy of K_5 or $K_{3,3}$ (possibly with other nodes along the edges shown)

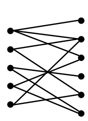
The Four-Color Theorem

Every planar graph is 4-colorable

(Appel & Haken, 1976)

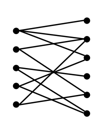
Bipartite Graphs

- A directed or undirected graph is *bipartite* if the vertices can be partitioned into two sets such that all edges go between the two sets
- Can represent a relationship between two sets, e.g. “is taking” between students and classes

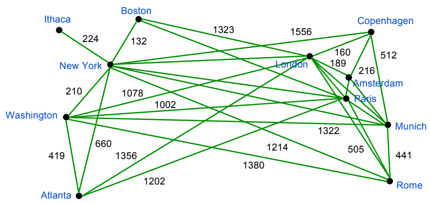


Bipartite Graphs

- The following are equivalent
  - G is bipartite
  - G is 2-colorable
  - G has no cycles of odd length



Traveling Salesperson



Find a path of minimum distance that visits every city

Implementing Digraphs

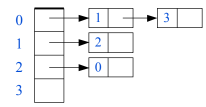
- Adjacency Matrix

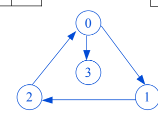
  g[u][v] is true iff there is an edge from u to v

	0	1	2	3
0		T		T
1			T	
2	T			
3				

- Adjacency List

  The list for u contains v iff there is an edge from u to v





Implementing Weighted Digraphs

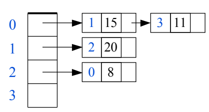
- Adjacency Matrix

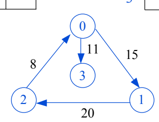
  g[u][v] is c iff there is an edge of cost c from u to v

	0	1	2	3
0		15		11
1			20	
2	8			
3				

- Adjacency List

  The list for u contains v,c iff there is an edge from u to v that has cost c





Implementing Undirected Graphs

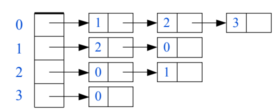
- Adjacency Matrix

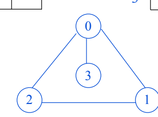
  g[u][v] is true iff there is an edge from u to v

	0	1	2	3
0		T	T	T
1	T		T	
2	T	T		
3	T			

- Adjacency List

  The list for u contains v iff there is an edge from u to v





Adjacency Matrix or Adjacency List?

n = number of vertices

m = number of edges

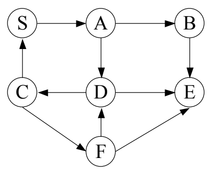
m_u = number of edges leaving u

- Adjacency Matrix
  - Uses space $O(n^2)$
  - Can iterate over all edges in time $O(n^2)$
  - Can answer "Is there an edge from u to v?" in O(1) time
  - Better for *dense* (i.e., lots of edges) graphs
- Adjacency List
  - Uses space O(m+n)
  - Can iterate over all edges in time O(m+n)
  - Can answer "Is there an edge from u to v?" in $O(m_u)$ time
  - Better for *sparse* (i.e., fewer edges) graphs

Goal: Find Shortest Path in a Graph

- Finding the shortest (min-cost) path in a graph is a problem that occurs often
  - Find the least-cost route between Ithaca and Detroit
  - Result depends on our notion of cost
    - least mileage
    - least time
    - cheapest
    - least boring
  - All of these "costs" can be represented as edge costs on a graph
- How do we find a shortest path?

Shortest Paths for Unweighted Graphs




```
bfsDistance(s):
    // s is the start vertex
    // dist[v] is length of s-to-v path
    // Initially dist[v] = ∞ for all v
    dist[s] = 0;
    Q.insert(s);

    while (Q nonempty) {
        v = Q.get();
        for (each w adjacent to v) {
            if (dist[w] == ∞) {
                dist[w] = dist[v]+1;
                Q.insert(w);
            }
        }
    }
```

Analysis for bfsDistance

- How many times can a vertex be placed in the queue?
- How much time for the for-loop?
  - Depends on representation
    - Adjacency Matrix: O(n)
    - Adjacency List: O(m_v)
- Time:
  - O(n^2) for adj matrix
  - O(m+n) for adj list

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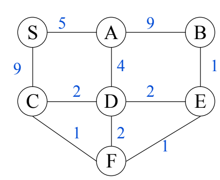
```

If There are Edge Costs?

- Idea #1
  - Add false nodes so that all edge costs are 1
  - But what if edge costs are large?
  - What if the costs aren't integers?
- Idea #2
  - Nothing "interesting" happens at the false nodes
    - Can't we just jump ahead to the next "real" node
  - Rule: always do the closest (real) node first
  - Use the array dist[ ] to
    - Report answers
    - *Keep track of what to do next*

Dijkstra's Algorithm

- Intuition
  - Edges are threads; vertices are beads
  - Pick up at s; mark each node as it leave the table
- Note: Negative edge-costs are *not allowed*



- s is the start vertex
- c(i,j) is the cost from i to j
- Initially, vertices are unmarked
- dist[v] is length of s-to-v path
- Initially, dist[v] = ∞, for all v

```
dijkstra(s):
    dist[s] = 0;
    while (some vertices are unmarked) {
        v = unmarked vertex with
            smallest dist;
        Mark v;
        for (each w adj to v) {
            dist[w] = min
                ( dist[w], dist[v] + c(v,w) );
        }
    }
```

Dijkstra's Algorithm using Adj Matrix

- While-loop is done n times
- Within the loop
  - Choosing v takes O(n) time
    - Could do this faster using PQ, but no reason to
  - For-loop takes O(n) time
- Total time = O(n^2)

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Dijkstra's Algorithm using Adj List

- Looks like we need a PQ
  - Problem: priorities are updated as algorithm runs
  - Can insert pair (v,dist[v]) in PQ whenever dist[v] is updated
  - At most m things in PQ
- Time O(n + m log m)
- Using a more complicated PQ (e.g., Pairing Heap), time can be brought down to O(m + n log n)

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Dijkstra's Algorithm for Digraphs

- Algorithm works on both undirected and directed graphs without modification
- As before: Negative edge-costs are *not allowed*

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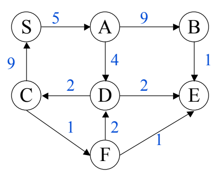


```
dijsktra(s):
    dist[s] = 0;
    while (some vertices are unmarked) {
        v = unmarked vertex with
            smallest dist;
        Mark v;
        for (each w adj to v) {
            dist[w] = min
                ( dist[w], dist[v] + c(v,w) );
        }
    }
```

Greedy Algorithms

- Dijkstra's Algorithm is an example of a *Greedy Algorithm*
- The Greedy Strategy is an algorithm design technique
  - Like Divide & Conquer
- The Greedy Strategy is used to solve optimization problems
  - The goal is to find the best solution
- Works when the problem has the *greedy-choice property*
  - A global optimum can be reached by making locally optimum choices

- Problem: Given an amount of money, find the smallest number of coins to make that amount
- Solution: Use a Greedy Algorithm
  - Give as many large coins as you can
- This greedy strategy produces the optimum number of coins for the US coin system
- Different money system ⇒ greedy strategy may fail
  - For example: suppose the US introduces a 4¢ coin