Lower bounds on sorting & Standard collection ADTs

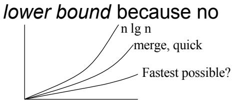
Lecture 19
CS211 Spring 06

Sorting algorithm summary

- The ones we have discussed
  - Insertion Sort
  - Selection Sort
  - Merge Sort
  - Quick Sort
- Other sorting algorithms
  - Heap Sort (uses priority queue)
  - Shell Sort (in text)
  - Bubble Sort (nice name, slow)
  - Radix Sort
  - Bin Sort
  - Counting Sort
- Why so many?
  - Stable sorts: *Ins, Sel, Mer*
  - Worst-case O(n log n): *Mer, Hea*
  - Expected-case O(n log n): *Mer, Hea, Qui*
  - Best for nearly-sorted sets: *Ins*
  - No extra space needed: *Ins, Sel, Hea*
  - Fastest in practice: *Qui*
  - Fastest on uniform integer keys O(*n*): *Radix*
  - Least data movement: *Sel*

Problem complexity

- Asymptotically fastest sorting algorithms are O(*n* lg *n*)
  - *kn* lg *n* is an *upper bound* on run time (for some *k*)
  - Can we do better?
- Some problems have an intrinsic complexity -- no algorithm can do better
  - Complexity of a problem is a *lower bound* because no algorithm can run faster



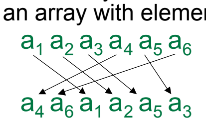
- What is the intrinsic complexity of sorting?

Lower bounds on sorting

- Goal: Determine the minimum time *required* to sort *n* items (no matter what order they come in)
  - Want *worst-case* time for the *best possible* algorithm
- Assumption: sorting algorithm works by comparing pairs of elements
  - in general: that's all you can do

The sorting job

- Sorting takes a array of *n* elements and produces an array with elements in sorted order

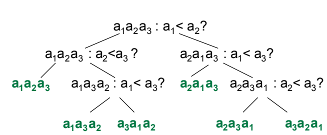


- Sorting finds a *permutation* of the original array
  - There are 1 × 2 × 3 × … × *n* = *n*! permutations
- Desired permutation is just the *inverse* permutation of the original ordering
- Any sorting algorithm must decide which of n! permutations will undo initial permutation

Comparison trees

- Any sorting algorithm performs some sequence of comparisons, depending on input:
  - Insertion sort on 2 3 1 4: 2<3? 3<1? 2<1? 3<4?
  - Insertion sort on 1 3 2 4: 1<3? 2<3? 1<2? 3<4?
- Comparisons form a *comparison tree*
  - At least *n*! leaves : every permutation must be a leaf
  - Worst-case # comparisons = height of tree

Comparison tree for insertion sort of three items



Time vs. Height

- Worst-case time for a sorting method must be ≥ the height of its comparison tree
  - The algorithm is doing more than just comparisons, but can use comparisons alone for lower bound
- Minimum possible height for a binary tree with *x* leaves is lg *x*
- With n! leaves?

  Height ≥ lg(n!) =
  lg(1 × 2 × … × n) =
  lg(1×n × 2×(n-1) × 3 × … × n/2)
  = lg(1 × n) + lg (2 × (n-1)) + …
  ≥ (n/2) * lg n
- **Any** comparison-based sorting algorithm **must** have a worst-case time of Ω(n lg n)
  - Lower bound; so we use big-Omega (Ω) instead of big-O

Using the Lower Bound on Sorting

Claim: I have a priority queue
- add time: O(1)
- removeMax time: O(1)

- True or false?

False (for general sets) because it could be used to sort in time O(*n*) using heapsort.

Heapsort: insert all elements into priority queue, extract in priority order.

Sorting in Linear Time

Several sorting methods take only linear time
- Counting Sort
  - Sorts integers from a small range: [0..k] where k = O(n)
- Radix Sort
  - The method used by card-sorters
  - Sorting time O(dn) where d is the number of “digits”
- Others…

- How do they get around the Ω(n lg n) lower bound?
  - Don’t use comparisons: use keys as numbers to index into arrays

Collection ADTs

- What are the useful abstractions for organizing data in collections?
  - So far: sets, priority queues
- How can they be implemented efficiently?
  - So far: lists, arrays, trees
- This lecture: more useful abstractions (but not how to implement them).

Set abstractions

```
class Set<T> {
   boolean contains(T elem);
}
```

- Mutable sets: elements can be added and removed from set (the usual approach)

  `void add(T elem)`
- Immutable sets: sets don’t change. Insertion or union produce new sets

  `Set add(T elem)`
  `Set union(Set<T> s)`
  - Implementable with data structures that share data (e.g., lists, trees), useful when related sets must coexist.

Set abstractions, cont’d

- Unordered sets:
  - No ordering on elements
  - Iterator produces elements in no particular order
  - No way to get from one element to next or previous
  - The basic abstraction & the most efficient approach if you don’t need ordering
- Ordered sets:
  - Elements are (abstractly!) kept in sorted order, can be iterated in order.
  - May be able to search within a range
  - May be able to find next or previous element in order
  - Useful if elements have natural ordering (e.g., dates)
  - Usually implemented as trees
- Bags (multisets): can contain same element more than once

Map abstractions

```
class Map<K, V> {
  V get(K key);
}
```

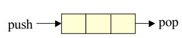
- Maintains an association between keys and values
- Every key occurs only once
- Can look up value associated with a key
- Also known as **associative arrays**, **dictionaries** (esp. with string keys)
- Java interface: java.util.Map

Map varieties

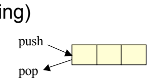
- Mutable maps
  - `void put(K key, V value)`
- Immutable maps
  - `Map put(K key, V value)  // non-destructive`
  - Unusual, implementable as tree
- Ordered maps: mappings are ordered by keys
  - Can view a map a set of (key, value) pairs where two pairs are considered “equal” or “less than” if their keys are.
  - Implemented as a tree with key and value at each node.
- Can use as an index.
  - Example: Collection of employee records might be a set of objects.
    - Might also have several maps as indices: from employee name to record object, from employee number to record object, …

Queues

- Queues contain elements but do not support random lookup
- FIFO queues



  - Push at one end, pop at the other
  - Helpful for delaying work (buffering)
- LIFO queues (stacks)



  - Push and pop from top
  - Good for saving and restoring state
- Priority queues
  - Elements have priority, are popped in priority order.

Summary

- Several useful abstractions for organizing, finding, updating information
- Choose the right abstraction for your program
  - May use several abstractions together
- Next: how to implement unordered sets and maps efficiently