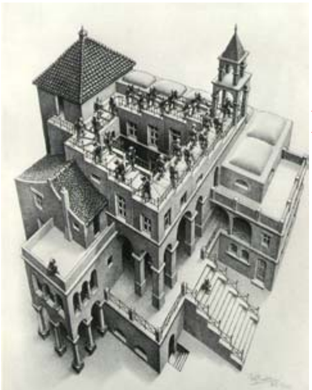


More on ADTs: Priority Queues & Dictionaries

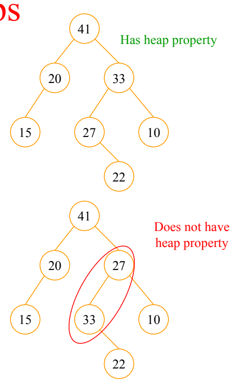
Lecture 17
CS211 – Fall 2005

Announcements

- Assignment 5 is online (since Friday)
  - Due Wednesday, Nov 2
  - Multiple small tasks

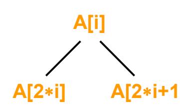
Heaps

- A heap is a tree that
  - Has a particular shape (we'll come back to this) and
  - Has the *heap property*
- *Heap property*
  - Each node's value (its *priority*) is ≤ the value of its parent
  - This version is for a max-heap (max value at the root)
    - There is a similar heap property for a min-heap (min at the root)



Heap Implementation (the Big Trick)

- Can avoid using pointers!



- Use a complete binary tree stored in an array
  - Definition: *Complete* means that each level of the tree is filled except possibly the last, which is filled from left to right
- For A[i]
  - left child = 2 * i
  - right child = 2 * i + 1
  - parent = ⌊i / 2⌋

Insert and GetMax Pseudocode

```
insert (Item):
   Place item in a leaf (= next empty position in array);
   while (item > parent) {Swap item with parent;}        // BubbleUp

getMax ():
   max = root.value;
   Swap root and last item (call it v) in heap; // Ensures same shape for heap
   Decrease heap size by 1 (i.e., access less of the array);
   while (v < one of its children)              // BubbleDown
         {Swap v with its largest child;}
   return max;
```

To Build a Heap

- How long to construct a heap, given the items?
- Worst-case time for insert() is O(log n)
- Total time to build heap using insert() is
  O(log 1) + O(log 2) + ... + O(log n)
  or O(n log n)

Can we do better?

- We had two heap-fixing methods
  - bubbleUp: move up the tree as long as we're > our parent
  - bubbleDown: move down the tree as long as we're < one of our children
- If we build the heap from the bottom-up using bubbleDown then we can build it in time O(n) (Wow!)

Efficient Heap Building

- Build from the bottom-up
- If there are n items in the heap then...
  - There are about n/2 mini-heaps of height 1
  - There are about n/4 mini-heaps of height 2
  - There are about n/8 mini-heaps of height 3 and so on
- The time to fix up a mini-heap is O(its height)
- Total time spent fixing heaps is thus bounded by
  $n/2 + 2n/4 + 3n/8 + \ldots$
- This can be rewritten as
  $n(1/2 + 2/4 + \ldots + i/2^i + \ldots)$
  $= n(2)$
- Thus total heap-building time (using the bottom-up method) is O(n)

HeapSort

- Given a Comparable[ ] array of length n,
  - Put all n elements into a heap: O(n) or O(n log n)
  - Repeatedly get the min: O(n log n)

```
public static void heapSort(Comparable[] a) {
   PriorityQueue<Comparable> pq = new PQ<Comparable>();
   for (Comparable x : a) { pq.put(x); }
   for (int i = 0; i < a.length; i++) { a[i] = pq.get(); }
}
```

PQ Application: Simulation

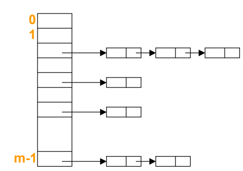
- Example: Given a probabilistic model of bank-customer arrival times and transaction times, how many tellers are needed
  - Assume we have a way to generate random inter-arrival times
  - Assume we have a way to generate transaction times
  - Can simulate the bank to get some idea of how long customers must wait

<u>Time-Driven Simulation</u>
- Check at each *tick* to see if any event occurs

<u>Event-Driven Simulation</u>
- Advance clock to next event, skipping intervening *ticks*
- This uses a PQ!

Another PQ Implementation

- If there are only a few possible priorities then can use an array of queues
  - Each array position represents a priority (0..m-1 where m is the array size)
  - Each queue holds all items that have that priority
- One text [Skiena] calls this a *bounded height priority queue*
- Time for insert: O(1)
- Time for getMax:
  - O(m) in the worst-case
  - Generally, faster
- Example: airline check-in



Other PQ Operations

<u>delete</u>
a particular item

<u>update</u>
an item (change its priority)

<u>join</u>
two priority queues

- For delete and update, we need to be able to find the item
  - One way to do this: Use a Dictionary to keep track of the item's position in the heap
- Efficient joining of 2 Priority Queues requires another data structure
  - Skew Heaps or Pairing Heaps (Chapter 23 in text)

Recall: Sets & Dictionaries

- ADT Set
  Operations:
  void insert (Object element);
  boolean contains (Object element);
  void remove (Object element);
  boolean isEmpty ( );
  void makeEmpty ( );
  - Note: no duplicates allowed
    - A "set" with duplicates is usually called a *bag*
- Where used:
  - Wide use within other algorithms

- ADT Dictionary
  Operations:
  void insert (Object key, Object value);
  void update (Object key, Object value);
  Object find (Object key);
  void remove (Object key);
  boolean isEmpty ( );
  void makeEmpty ( );
  - Think of
    key = word; value = definition
- Where used:
  - Symbol tables
  - Wide use within other algorithms

Implementing Sets

- Recall: ADT Set
  Operations:
  void insert (Object element);
  boolean contains (Object element);
  void remove (Object element);
  boolean isEmpty ( );
  void makeEmpty ( );
- Can use a Dictionary
  - Values in (key, value) pairs are ignored
  - All operations are expected time O(1) using hash table (see next several slides)
- *If* the universe is not too large
  - Can use a table of bits (i.e., a bit-vector)
    - We need n bits for a universe of size n
  - This implementation also allows for fast *union*, *intersection*, and *complement*

Goal: Design a *Dictionary*

- Operations
  - void insert (key,value)
  - void update (key, value)
  - Object find (key)
  - void remove (key)

Array implementation:

Uses an array of (key,value) pairs

	Unsorted	Sorted
insert	O(1)	O(n)
update	O(n)	O(log n)
find	O(n)	O(log n)
remove	O(n)	O(n)

n is the number of items currently held in the array

Direct Address Table

- Assumes the key set is from a small *Universe*
- Example: Addresses on my street
  - Start at 1, go to 40
  - A few lots don't have houses
- For a *Direct Address Table*, we make an array as large as the *Universe*
- To find an entry, we just index to that entry of the array
- Dictionary operations all take O(1) time

What if the Universe is large?

- Idea is to re-use table entries via a *hash function* h
- h: U → [0,…,m-1] where m = table size
- h must
  - Be easy to compute
  - Cause few *collisions*
  - Have equal probability for each table position

Typical situation:

U = all legal identifiers

Typical hash function:

h converts each letter to a number and we compute a function of these numbers

A Hashing Example

- Suppose each word below has the following hashCode

jan	7
feb	0
mar	5
apr	2
may	4
jun	7
jul	3
aug	7
sep	2
oct	5

- How do we resolve collisions?
  - We'll use *chaining*: each table position is the head of a list
  - For any particular problem, this *might* work terribly
- In practice, using a good hash function, we can assume each position is equally likely

Analysis for Hashing with Chaining

- Analyzed in terms of *load factor* λ = n/m = (items in table)/(table size)
- We count the expected number of *probes* (key comparisons)
- Goal: Determine U = number of probes for an *unsuccessful* search
- Claim U is the same as the average number of items per table position = n/m = λ
- Claim S = number of probes for a *successful* search = 1 + λ/2

Table Doubling

- We know each operation takes time $O(\lambda)$ where $\lambda = n/m$
- But isn't $\lambda = \Theta(n)$?
- What's the deal here? It's still linear time!

Table Doubling:

- Set a bound for λ (call it λ_0)
- Whenever λ reaches this bound we
  - Create a new table, twice as big and
  - Re-insert all the data
- Easy to see operations *usually* take time O(1)
  - But sometimes we copy the whole table

Analysis of Table Doubling

- Suppose we reach a state with n items in a table of size m and that we have just completed a table doubling

	Copying Work
Everything has just been copied	n inserts
Half were copied previously	n/2 inserts
Half of those were copied previously	n/4 inserts
…	…
Total work	n + n/2 + n/4 + … = 2n

Analysis of Table Doubling, Cont'd

- Total number of insert operations needed to reach current table = copying work + initial insertions of items
  = 2n + n = 3n inserts
- Each insert takes expected time $O(\lambda_0)$ or O(1), so total expected time to build entire table is O(n)
- Thus, expected time per operation is O(1)

- Disadvantages of table doubling:
  - Worst-case insertion time of O(n) is definitely achieved (but rarely)
  - Thus, not appropriate for time critical operations

Java Hash Functions

- Most Java classes implement the hashCode() method
- hashCode() returns an *int*
- Java's HashMap class uses h(X) = X.hashCode() mod m
- h(X) in detail:

```
int hash = X.hashCode();
int index = (hash & 0x7FFFFFFF) % m;
```

What hashCode() returns:

Integer: uses the int value

Float: converts to a bit representation and treats it as an int

Short **Strings**: 37*previous + value of next character

Long **Strings**: sample of 8 characters; 39*previous + next value

hashCode( ) Requirements

- Contract for hashCode() method:
  - Whenever it is invoked in the same object, it must return the same result
  - Two objects that are equal must have the same hash code
  - Two objects that are not equal should return different hash codes, but are not required to do so

Hash Tables in Java

java.util.HashMap
java.util.HashSet
java.util.Hashtable (legacy)

- Use chaining
- Initial (default) size = 101
- Load factor = $\lambda_0 = 0.75$
- Uses table doubling (2*previous+1)

A node in each *chain* looks like this:

hashCode	key	value	next

original hashCode (before mod m)
[Allows faster rehashing and (possibly) faster key comparison]

Hashing Application: Spell Checking

- We want to create a "spelling dictionary" containing 10,000 words
  - A spelling query should be fast
  - Should return true iff word is contained in dictionary
- Basic idea:
  - Use a Hashtable consisting only of bits (say 100K bytes or about 800,000 bits)
  - Compute a hash value for each word and turn on the corresponding bit in the table
  - What's the probability of a false positive? (It's too high!)
  - Fix: Use more hash functions

Linear & Quadratic Probing

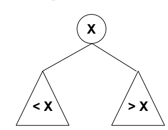
- These are techniques in which all data is stored directly within the hash table array
- Linear Probing
  - Probe at h(X), then at
    - h(X) + 1
    - h(X) + 2
    - …
    - h(X) + i
  - Leads to *primary clustering*
    - Long sequences of filled cells
- Quadratic Probing
  - Similar to Linear Probing in that data is stored within the table
  - Probe at h(X), then at
    - h(X)+1
    - h(X)+4
    - h(X)+9
    - …
    - h(X)+ i^2
  - Works well when
    - $\lambda < 0.5$
    - table size is *prime*

Hash Table Pitfalls

- Good hash function is *required*
- Watch the load factor (λ), especially for Linear & Quadratic Probing

Dictionary Implementations

- Ordered Array
  - Better than unordered array because Binary Search can be used
- Unordered Linked-List
  - Ordering doesn't help
- Direct Address Table
  - Small universe ⇒ limited usage
- Hashtables
  - O(1) expected time for Dictionary operations
- Goal: Want ability to *report-in-order*, but can't afford inefficiency of ordered array
- Idea: Use a Binary Search Tree (BST)
- BST Property:



Deleting from a BST

Cases:

- Delete a leaf
  - easy
- Delete a node with just one child
  - delete and replace with child
- Delete a node with two children
  - delete node's successor
  - write successor's data into node
- How do we find the successor?
- The successor always has at most one child. Why?
- Would work just as well using predecessor instead of successor

BST Performance

- Time for insert(), find(), update(), remove() is O(*h*) where *h* is the height of the tree
- How bad can *h* be?
- Operations are fast if tree is *balanced*
- How balanced is a random tree?
  - If items are inserted in random order then the expected height of a BST is O(log *n*) where *n* is the number of items
- If deletion is allowed
  - Tree is no longer random
  - Tree is likely to become unbalanced

Analysis Sketch for Random BST

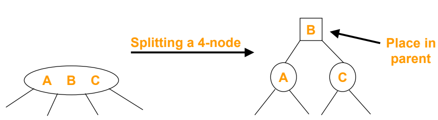
- Only the number of items and their order is important
  - Can restrict our attention to BSTs containing items $\{1,\ldots, n\}$
- We assume that each item is equally likely to appear as the root
- Define $H(n) \equiv$ *expected* height of BST of size n
- If item i is the root then expected height is
  $1 + \max \{ H(i-1), H(n-i) \}$
  We average this over all possible i
- Can solve the resulting recurrence (by induction) to show
  $H(n) = O(\log n)$

Why use a BST instead of a Hashtable?

- If we use a *balanced* BST scheme then we achieve guaranteed *worst-case* time bound of O(log n) for typical Dictionary ops
- There are some operations that can be efficient on BSTs, but very inefficient on Hashtables
  - report-elements-in-order
  - getMin
  - getMax
  - select(k) // find the *k*-th element
    - (maintain size of each subtree by using an additional *size* field in each node)
- Note that balanced BST schemes can be difficult to implement
  - But there are lots of reliable codes for these schemes available on the Web
  - Java includes a balanced BST scheme among its standard packages (java.util.TreeMap and java.util.TreeSet)

Example Balancing Scheme: 234-Trees

- Nodes have 2, 3, or 4 children (and contain 1, 2, or 3 keys, respectively)
- All leaves are at the same level
- Basic rule for insertion: We hate 4-nodes
  - Split a 4-node whenever you find one while coming down the tree
  - Note: this requires that parent is not a 4-node
- Delete is harder than insert
  - For delete, we hate 2-nodes
  - As in BSTs, cannot delete from a nonleaf so we use same BST trick: delete successor and recopy its data



234-Tree Analysis

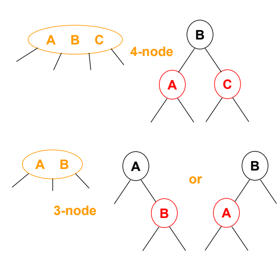
- Time for insert or get is proportional to tree's height
- How big is tree's height h?
- Let n be the number of nodes in a tree of height h
  - n is large if all nodes are 4-nodes
  - n is small if all nodes are 2-nodes
- Can use this to show
  $h = O(\log n)$

Analysis of tree height:
- Let N be the number of nodes, n be the number of items, and h be the height
- Define h so that a tree consisting of a single node is height 0
- It's easy to see $1+2+4+\ldots+2^h \leq N \leq 1+4+16+\ldots+4^h$
- It's also easy to see $N \leq n \leq 3N$
- Using the above, we have $n \geq 1+2+4+\ldots+2^h = 2^{h+1}-1$
- Rewriting, we have $h \leq \log(n+1) - 1$ or $h = O(\log n)$
- Thus, Dictionary operations on 234-trees take time O(log n) in the worst case

234-Tree Implementation

- Can implement all nodes as 4-nodes
  - Wasted space
- Can allow various node sizes
  - Requires recopying of data whenever a node changes size
- Can use BST nodes to emulate 2-, 3-, or 4-nodes

Using BSTs to Emulate 234-Trees



- A 2-node can be represented with a standard BST node
- A 4-node can be represented with three BST nodes
- A 3-node can be represented with two BST nodes (in two different ways)

Red-Black Trees

- We need a way to tell when an emulated 234-node starts and ends
- We mark the nodes
  - Black: "root" of 234-node
  - Red: belongs to parent
  - Requires one bit per node
- 234-tree rules become rules for *rotations* and color changes in red-black trees
- Result:
  - one black node per 234-node
  - Number of black nodes on path from root to leaf is same as height of 234-tree
  - All paths from root to leaf have same number of black nodes
  - On any path: at most one red node per black node
  - Thus tree height for red-black tree is O(log n)

Balanced Tree Schemes

- AVL trees [1962]
  - named for initials of Russian creators
  - uses rotations to ensure heights of child trees differ by at most 1
- 23-Trees [Hopcroft 1970]
  - similar to 234-tree, but repairs have to move back up the tree
- B-Trees [Bayer & McCreight 1972]
- Red-Black Trees [Bayer 1972]
  - not the original name
- Red-black convention & relation to 234-trees [Guibas & Stolfi 1978]
- Splay Trees [Sleator & Tarjan 1983]
- Skip Lists [Pugh 1990]
  - developed at Cornell

Selecting a Dictionary Scheme

- Use an unordered array for small sets (< 20 or so)
- Use a Hash Table if possible
  - Cannot efficiently do some ops that are easy with BSTs
  - Running times are expected rather than worst-case
- Use an ordered array if few changes after initialization
- B-Trees are best for large data sets, external storage
  - Widely used within data base software
- Otherwise, Red-Black Trees are current scheme of choice
- Skip Lists are supposed to be easier to implement
  - But shouldn't have to implement—use existing code
- Splay trees are useful if some items are accessed more often than others
  - But if you know which items are most-commonly accessed, use a separate data structure

Selecting a Priority Queue Scheme

- Use an unordered array for small sets (< 20 or so)
- Use a sorted array or sorted linked list if few insertions are expected
- Use an array of linked lists if there are few priorities
  - Each linked list is a queue of equal-priority items
  - Very easy to implement
- Otherwise, use a Heap if you can
- Heap + Hashtable
  - Allow *change-priority* operation to be done in O(log n) expected time
- Balanced tree schemes
  - Useful and practical
- There are a number of alternate implementations that allow additional operations
  - Skew heaps
  - Pairing heaps
  - Fibonacci heaps
  - …