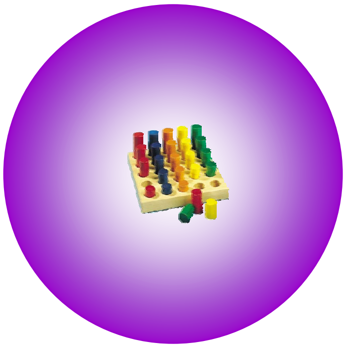


Bounds on Sorting

Lecture 14
CS211 – Fall 2005

Prelim Announcements

- Prelim 1
  - Tonight 7:30 – 9:00pm
  - Last names starting with **A-F** are in HO 110
  - Last names starting with **G-Z** are in HO B14
- Grades will be available tomorrow (Friday)
  - This is the last day to drop a course
- Office hours are available before the exam
  - Regularly scheduled
    - 11:00 – 12:00
    - 1:25 – 2:15
    - 3:00 – 4:00
  - Extra
    - 12:20 – 1:25
  - Check course website for latest info

More Announcements

- Using consultants
  - Do not work in consulting room after receiving help
    - Work somewhere else so other students can ask questions
  - Do not use consultants as "human compilers"
    - You are responsible for testing your code on your own
    - Not incrementally with a consultant
- ACSU Ultimate Coding Challenge
  - Prizes: Xbox or Portable Media Center
  - Saturday, October 15th, 2005 @ 10:00 am - 2:00 pm
  - Upson 319 (CSUG Lab)
  - FREE pizza and ACSU t-shirts for all participants!
  - See 211 website for more info

Sorting Algorithm Summary

- The ones we have discussed
  - Insertion Sort
  - Selection Sort
  - Merge Sort
  - Quick Sort
- Other sorting algorithms
  - Heap Sort (come back to this)
  - Shell Sort (in text)
  - Bubble Sort (nice name)
  - Radix Sort
  - Bin Sort
  - Counting Sort
- Why so many? Do Computer Scientists have some kind of sorting fetish or what?
  - Stable sorts: *Ins, Sel, Mer*
  - Worst-case O(n log n): *Mer, Hea*
  - Expected-case O(n log n): *Mer, Hea, Qui*
  - Best for nearly-sorted sets: *Ins*
  - No extra space needed: *Ins, Sel, Hea*
  - Fastest in practice: *Qui*
  - Least data movement: *Sel*

Programming Problem Strategies

- Goal: Make it easier to solve programming problems
- Basic Data Structures
  - I recognize this; I can use this well-known data structure
  - Examples: Stack, Queue, Priority Queue, Dictionary
- Algorithm Design Methods
  - I can design an algorithm to solve this
  - Examples: Divide & Conquer, Greedy, Dynamic Programming
- Problem Reductions
  - I can change this problem into another with a known solution
  - Or, I can show that a reasonable algorithm is most-likely impossible
  - Examples: reduction to network flow, NP-complete problems

Recall: Analysis of MergeSort

- Time for Merge is O(n) where n is the number of elements being merged
- Time for MergeSort

  $T(n) = 2T(n/2) + O(n)$
  and $T(1) = O(1)$

  Recurrence can be simplified
  to $T(n) = 2T(n/2) + n$

  Solution is $T(n) = O(n \log n)$

- One solution method for this recurrence

  Can divide by n to get
  $T(n)/n = T(n/2)/(n/2) + 1$

  Define $S(n) = T(n)/n$

  $S(n) = S(n/2) + 1$

  Easy to see that
  $S(n) = 2 + \log n$

  Thus $T(n) = n(2 + \log n)$ or
  $T(n) = O(n \log n)$

Solving Recurrences

Recurrences are important when using Divide & Conquer to design an algorithm

Solution techniques:

- Can sometimes change variables to make it into a simpler recurrence
- Make a guess then prove the guess correct by induction
- Build a recursion tree and use it to determine solution
- Can use the *Master Method*
  - A "cookbook" scheme that handles many common recurrences

To solve $T(n) = aT(n/b) + f(n)$ compare $f(n)$ with $n^{\log_b a}$

- Solution is $T(n) = O(f(n))$ if f(n) grows more rapidly
- Solution is $T(n) = O(n^{\log_b a})$ if $n^{\log_b a}$ grows more rapidly
- Solution is $T(n) = O(f(n) \log n)$ if both grow at same rate

- Not an exact statement of the theorem [f(n) must be "well-behaved"]
- See text for a similar theorem

Recurrence Relation Examples

- $T(n) = T(n-1) + 1$ [Linear Search]
  - $T(n) = O(n)$
- $T(n) = T(n-1) + n$ [QuickSort worst-case]
  - $T(n) = O(n^2)$
- $T(n) = T(n/2) + 1$ [Binary Search]
  - $T(n) = O(\log n)$
- $T(n) = T(n/2) + n$
  - $T(n) = O(n)$
- $T(n) = 2\,T(n/2) + n$ [MergeSort]
  - $T(n) = O(n \log n)$

Recurrences & CS211

- Solving recurrences is like integration
  - No general techniques work for all recurrences
- For CS 211, we just expect you to remember a few common patterns

Lower Bounds on Sorting: Goals

- Goal: Determine the minimum time *required* to sort *n* items
- Note: we want *worst-case* not *best-case* time
  - Best-case doesn't tell us much; for example, we know Insertion Sort takes O(n) time on already-sorted input
  - We want to determine the *worst-case* time for the *best-possible* algorithm

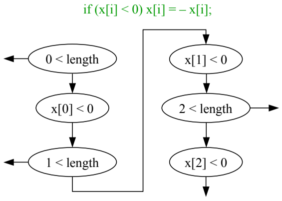
- But how can we prove anything about the *best possible* algorithm?
  - We want to find characteristics that are common to *all* sorting algorithms
  - Let's try looking at *comparisons*

Comparison Trees

- Any algorithm can be "unrolled" to show the comparisons that are (potentially) performed

Example

```
for (int i = 0; i < x.length; i++)
    if (x[i] < 0) x[i] = – x[i];
```



- In general, you get a *comparison tree*
- If the algorithm fails to terminate for some input then the comparison tree is infinite
- The height of the comparison tree represents the *worst-case number of comparisons* for that algorithm

Lower Bounds on Sorting: Notation

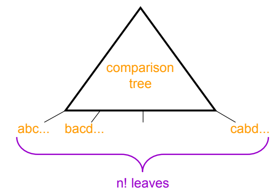
- Suppose we want to sort the items in the array B[ ]
- Let's name the items
  - a_1 is the item initially residing in B[1], a_2 is the item initially residing in B[2], etc.
  - In general, a_i is the item initially stored in B[i]
- Rule: an item keeps its name forever, but it can change its location
  - Example: after swap(B,1,5), a_1 is stored in B[5] and a_5 is stored in B[1]

The *Answer* to a Sorting Problem

- An *answer* for a sorting problem tells where each of the a_i resides when the algorithm finishes
- How many answers are possible?
- The *correct* answer depends on the actual values represented by each a_i
- Since we don't know what the a_i are going to be, it has to be *possible* to produce each permutation of the a_i
- For a sorting algorithm to be valid it must be possible for that algorithm to give any of n! potential answers

Comparison Tree for Sorting

- Every sorting algorithm has a corresponding *comparison tree*
  - Note that other stuff happens during the sorting algorithm, we just aren't showing it in the tree
- The comparison tree must have n! (or more) leaves because a valid sorting algorithm must be able to get any of n! possible answers
- Comparison tree for sorting n items:



Time vs. Height

- The worst-case time for a sorting method must be ≥ the height of its comparison tree
  - The height corresponds to the worst-case number of comparisons
  - Each comparison takes $\Theta(1)$ time
  - The algorithm is doing more than just comparisons
- What is the minimum possible height for a binary tree with n! leaves?

  Height $\geq \log(n!) = \Theta(n \log n)$
- This implies that <u>any</u> comparison-based sorting algorithm <u>must</u> have a worst-case time of $\Omega(n \log n)$
  - Note: this is a lower bound; thus, the use of big-Omega instead of big-O

Using the Lower Bound on Sorting

<u>Claim</u>: I have a PQ
- Insert time: O(1)
- GetMax time: O(1)

- True or false?

False (for general sets) because if such a PQ existed, it could be used to sort in time O(n)

<u>Claim</u>: I have a PQ
- Insert time: O(loglog n)
- GetMax time: O(loglog n)

- True or false?

False (for general sets) because it could be used to sort in time O(n loglog n)

True for items with priorities in range 1..n [van Emde Boas] (Note: such a set can be sorted in O(n) time)

Sorting in Linear Time

There are several sorting methods that take linear time
- Counting Sort
  - Sorts integers from a small range: [0..k] where k = O(n)
- Radix Sort
  - The method used by the old card-sorters
  - Sorting time O(dn) where d is the number of "digits"
- Others…

- How do these methods get around the $\Omega(n \log n)$ lower bound?
  - They don't use comparisons

Best Sorting Method?

- What sorting method works best?
  - QuickSort is best general-purpose sort
    - But it's not *stable*
  - MergeSort is a good choice if you need a *stable* sort
  - Counting Sort or Radix Sort can be best for *some* kinds of data