Sorting

Lecture 13
CS211 – Fall 2005

InsertionSort

```
// Code for sorting a[ ], an array of int
for (int i = 1; i < a.length; i++) {
    int temp = a[ i ];
    int k = i;
    for (; 0 < k && temp < a[k–1]; k – –)
            a[k] = a[k–1];
    a[k] = temp;
}
```

- Many people sort cards this way
- Invariant: everything to left of i is already sorted
- Works especially well when input is *nearly sorted*

- Runtime
  - Worst-case
    - $O(n^2)$
    - Consider reverse-sorted input
  - Best-case
    - $O(n)$
    - Consider sorted input
  - Expected-case
    - $O(n^2)$
    - Can count expected number of inversions
      - Pair a sequence with its reverse
      - The average number of inversions is n(n-1)/4
      - See text

SelectionSort

- To sort an array of size n:
  - Examine all elements from 0 to (n-1); find the smallest one and swap it with the 0^{th} element of the array
  - Examine all elements from 1 to (n-1); find the smallest in that part of the array and swap it with the 1^{st} element of the array
  - In general, at the i^{th} step, examine array elements from i to (n-1); find the smallest element in that range, and exchange it with the i^{th} element of the array
- This is the other common way for people to sort cards
- Runtime
  - Worst-case
    - $O(n^2)$
  - Best-case
    - $O(n^2)$
  - Expected-case
    - $O(n^2)$

Divide & Conquer?

- It often pays to
  1) break the problem into smaller subproblems,
  2) solve the subproblems separately, and then
  3) assemble a final solution
- This technique is called *Divide-and-Conquer*
- Caveat: the partitioning and assembly processes cannot be too expensive
- Can we apply this approach to sorting?

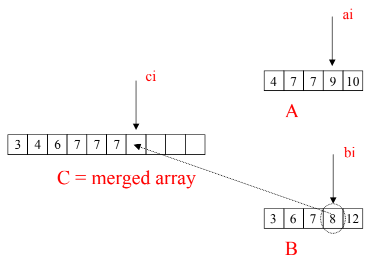
MergeSort

- Quintessential divide-and-conquer algorithm
- Divide array into equal parts, sort each part, then merge
- Three questions:
  - Q1: How do we divide array into two equal parts?
  - A1: Use indices into array
  - Q2: How do we sort the parts?
  - A2: call MergeSort recursively!
  - Q3: How do we merge the sorted subarrays?
  - A3: Have to write some (easy) code

Merging Sorted Arrays A and B

- Create an array C of size = size of A + size of B
- Keep three indices:
  - ai into A
  - bi into B
  - ci into C
- Initialize all three indices to 0 (start of each array)
- Compare element A[ai] with B[bi], and move the smaller element into C[ci]
- Increment the appropriate indices (ai or bi), and ci
- If either A or B is empty, copy remaining elements from the other array (B or A, respectively) into C

Merging Sorted Arrays



MergeSort Analysis

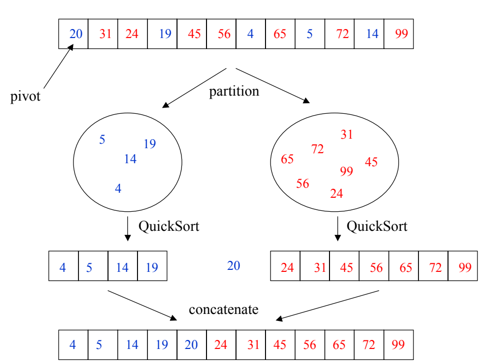
- Outline (text has detailed code)
  - Split array into two halves
  - Recursively sort each half
  - Merge the two halves
- Merge = combine two sorted arrays to make a single sorted array
  - Rule: Always choose the smallest item
  - Time: O(n) where n is the combined size of the two arrays
- Runtime recurrence
  - Let T(n) be the time to sort an array of size n
  - $T(n) \le 2T(n/2) + cn$
  - $T(1) = c$
- Can show by induction that $T(n) = O(n \log n)$
- Alternately, can show $T(n) = O(n \log n)$ by looking at tree of recursive calls

MergeSort Notes

- Asymptotic complexity: O(n log n)
  - *Much* faster than $O(n^2)$
- Disadvantage
  - Need extra storage for temporary arrays
  - In practice, this can be a serious disadvantage, even though MergeSort is *asymptotically optimal for sorting*
  - Can do MergeSort in place, but this is *very* tricky (and it slows down the algorithm significantly)
- Are there good sorting algorithms that do not use so much extra storage?
  - Yes: QuickSort

QuickSort

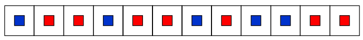
- Intuitive idea
  - Given an array A to sort, choose a pivot value p
  - *Partition* A into two subarrays, AX and AY
    - AX contains only elements ≤ p
    - AY contains only elements ≥ p
  - Sort subarrays AX and AY separately
  - *Concatenate* (not merge!) sorted AX and AY to produce sorted A
    - Note that concatenation is easier than merging



QuickSort Questions

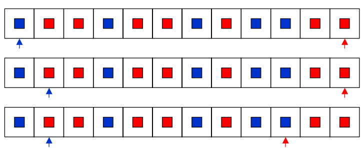
- Key problems
  - How should we choose a *pivot*?
  - How do we *partition* an array in place?
- Partitioning in place
  - Can be done in O(n) time
  - See next few slides
- Choosing a pivot
  - Ideal pivot is median since this splits array in half
  - Unfortunately, computing the median is expensive
  - Popular heuristics
    - Use first value in array as pivot (this is a *bad* choice)
    - Use middle value in array as pivot
    - Use median of first, last, and middle values in array as pivot

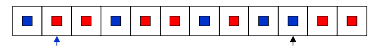
In-Place Partitioning



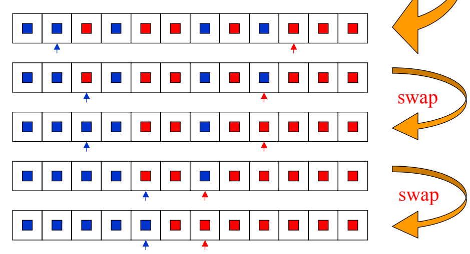
How can we move all the blues to the left of all the reds?

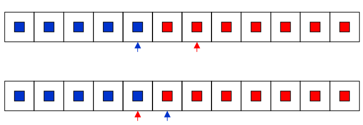
1. Keep two indices, LEFT and RIGHT
2. Initialize LEFT at start of array and RIGHT at end of array
3. Invariant: all elements to left of LEFT are blue
   all elements to right of RIGHT are red
4. Keep advancing indices until they pass, maintaining invariant





Now neither LEFT nor RIGHT can advance and maintain invariant. We can swap red and blue pointed to by LEFT and RIGHT indices. After swap, indices can continue to advance until next conflict.





- Once indices cross partitioning is done
- If you replace blue with ≤p and red with ≥p, this is exactly what we need for QuickSort partitioning
- Notice that after partitioning, array is partially sorted
- Recursive calls on partitioned subarrays will sort subarrays
- No need to copy/move arrays since we partitioned in place

QuickSort Analysis

- Runtime analysis (worst-case)
  - Partition can work badly producing this: | p | ≥ p |
  - Runtime recurrence
    - $T(n) = T(n-1) + n$
  - This can be solved to show worst-case $T(n) = O(n^2)$
- Runtime analysis (expected-case)
  - More complex recurrence (see text)
  - Can solve to show expected $T(n) = O(n \log n)$
- Can improve constant factor by avoiding QuickSort on small sets
  - Switch to InsertionSort (for example) for sets of size, say, 8 or less
  - Definition of *small* depends on language, machine, etc.

Sorting Algorithm Summary

- The ones we have discussed
  - Insertion Sort
  - Selection Sort
  - Merge Sort
  - Quick Sort
- Other sorting algorithms
  - Heap Sort (come back to this)
  - Shell Sort (in text)
  - Bubble Sort (nice name)
  - Radix Sort
  - Bin Sort
  - Counting Sort
- Why so many? Do Computer Scientists have some kind of sorting fetish or what?
  - Stable sorts: *Ins, Sel, Mer*
  - Worst-case O(n log n): *Mer, Hea*
  - Expected-case O(n log n): *Mer, Hea, Qui*
  - Best for nearly-sorted sets: *Ins*
  - No extra space needed: *Ins, Sel, Hea*
  - Fastest in practice: *Qui*
  - Least data movement: *Sel*

Lower Bounds on Sorting: Goals

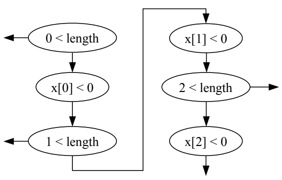
- Goal: Determine the minimum time *required* to sort *n* items
- Note: we want *worst-case* not *best-case* time
  - Best-case doesn't tell us much; for example, we know Insertion Sort takes O(n) time on already-sorted input
  - We want to determine the *worst-case* time for the *best-possible* algorithm
- But how can we prove anything about the *best possible* algorithm?
  - We want to find characteristics that are common to *all* sorting algorithms
  - Let's try looking at *comparisons*

Comparison Trees

- Any algorithm can be "unrolled" to show the comparisons that are (potentially) performed

Example

```
for (int i = 0; i < x.length; i++)
  if (x[i] < 0) x[i] = – x[i];
```



- In general, you get a *comparison tree*
- If the algorithm fails to terminate for some input then the comparison tree is infinite
- The height of the comparison tree represents the *worst-case number of comparisons* for that algorithm

Lower Bounds on Sorting: Notation

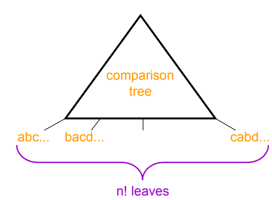
- Suppose we want to sort the items in the array B[ ]
- Let's name the items
  - a_1 is the item initially residing in B[1], a_2 is the item initially residing in B[2], etc.
  - In general, a_i is the item initially stored in B[i]
- Rule: an item keeps its name forever, but it can change its location
  - Example: after swap(B,1,5), a_1 is stored in B[5] and a_5 is stored in B[1]

The *Answer* to a Sorting Problem

- An *answer* for a sorting problem tells where each of the a_i resides when the algorithm finishes
- How many answers are possible?
- The *correct* answer depends on the actual values represented by each a_i
- Since we don't know what the a_i are going to be, it has to be *possible* to produce each permutation of the a_i
- For a sorting algorithm to be valid it must be possible for that algorithm to give any of n! potential answers

Comparison Tree for Sorting

- Every sorting algorithm has a corresponding *comparison tree*
  - Note that other stuff happens during the sorting algorithm, we just aren't showing it in the tree
- The comparison tree must have n! (or more) leaves because a valid sorting algorithm must be able to get any of n! possible answers
- Comparison tree for sorting n items:



Time vs. Height

- The worst-case time for a sorting method must be ≥ the height of its comparison tree
  - The height corresponds to the worst-case number of comparisons
  - Each comparison takes Θ(1) time
  - The algorithm is doing more than just comparisons
- What is the minimum possible height for a binary tree with n! leaves?

  Height ≥ log(n!) = Θ(n log n)

- This implies that <u>any</u> comparison-based sorting algorithm <u>must</u> have a worst-case time of Ω(n log n)
  - Note: this is a lower bound; thus, the use of big-Omega instead of big-O

Using the Lower Bound on Sorting

<u>Claim</u>: I have a PQ
- Insert time: O(1)
- GetMax time: O(1)
- True or false?

False (for general sets) because if such a PQ existed, it could be used to sort in time O(n)

<u>Claim</u>: I have a PQ
- Insert time: O(loglog n)
- GetMax time: O(loglog n)
- True or false?

False (for general sets) because it could be used to sort in time O(n loglog n)

True for items with priorities in range 1..n [van Emde Boas] (Note: such a set can be sorted in O(n) time)

Sorting in Linear Time

There are several sorting methods that take linear time

- Counting Sort
  - Sorts integers from a small range: [0..k] where k = O(n)
- Radix Sort
  - The method used by the old card-sorters
  - Sorting time O(dn) where d is the number of "digits"

- How do these methods get around the Ω(n log n) lower bound?
  - They don't use comparisons

- What sorting method works best?
  - QuickSort is best general-purpose sort (but it's not stable)
  - Counting Sort or Radix Sort can be best for *some* kinds of data