Searching & an Introduction to Asymptotic Complexity

Lecture 12
CS211 – Fall 2005

Announcements

- Prelim 1
  - Occurs at 7:30pm on Thursday (Oct 13) after Fall Break (i.e., 9 days from today)
  - Topics: all material from August & September
    - Includes Interfaces & Comparable
    - Not Searching & Sorting & Asymptotic Complexity (this week's topics)
- Exam conflicts
  - Email Kelly Patwell ASAP
  - We have a late-start exam (at 8:30pm) for those observing Yom Kippur
    - Email Kelly if you need to take the late-start exam
- Prelim 1 review sessions
  - Wed, Oct 12
  - Two identical sessions
    - 7:30 – 9:00pm
    - 9:00 – 10:30pm
  - See *Exams* on course website for more information
  - Individual appointments are available if you cannot attend the review sessions (email *one* TA to arrange appointment)
- Old exams are available for review on the course website
- Sections for Wed, Oct 12, are cancelled
  - This week's sections are the last before Prelim 1

What Makes a Good Algorithm?

- Suppose you have two possible algorithms or data structures that basically do the same thing; which is *better*?
- Well… what do we mean by *better*?
  - Faster?
  - Less space?
  - Easier to code?
  - Easier to maintain?
  - Required for homework?
- How do we measure time and space for an algorithm?

Sample Problem: Searching

- Determine if a *sorted* array of integers contains a given integer
- 1st solution: Linear Search (check each element)

```
static boolean find (int[ ] a, int item) {
  for (int i = 0; i < a.length; i++) {
        if (a[i] == item) return true;
        }
  return false;
  }
```

- 2nd solution: Binary Search

```
static boolean find (int[ ] a, int item) {
   int low = 0;
   int high = a.length - 1;
   while (low <= high) {
            int mid = (low+high)/2;
            if (a[mid] < item)
                     low = mid+1;
            else if (item < a[mid])
                     high = mid - 1;
            else return true;
            }
   return false;
   }
```

Linear Search vs. Binary Search

- Which one is better?
  - Linear Search is easier to program
  - But Binary Search is faster… isn't it?
- How do we measure to show that one is faster than the other
  - Experiment?
  - Proof?
  - But which inputs do we use?
- Simplifying assumption #1: Use the *size* of the input rather than the input itself
  - For our sample search problem, the input size is n+1 where n is the array size
- Simplifying assumption #2: Count the number of "*basic steps*" rather than computing exact times

One Basic Step = One Time Unit

- Basic step:
  - input or output of a scalar value
  - accessing the value of a scalar variable, array element, or field of an object
  - assignment to a variable, array element, or field of an object
  - a single arithmetic or logical operation
  - method invocation (not counting argument evaluation and execution of the method body)
- For a conditional, we count number of basic steps on the branch that is executed
- For a loop, we count number of basic steps in loop body times the number of iterations
- For a method, we count number of basic steps in method body (including steps needed to prepare stack-frame)

Runtime vs. Number of Basic Steps

- But isn't this cheating?
  - The runtime is not the same as the number of basic steps
  - Time per basic step varies depending on computer, on compiler, on details of code…
- Well… yes, it is cheating in a way
  - But the number of basic steps is *proportional* to the actual runtime
- Which is better?
  - n or n^2 time?
  - 100 n or n^2 time?
  - 10,000 n or n^2 time?
- As n gets large, multiplicative constants become less important
- Simplifying assumption #3: Multiplicative constants aren't important

Using Big-O to Hide Constants

- Roughly, $f(n) = O(g(n))$ means that f(n) grows like g(n) or slower

Definition: O(g(n)) is a *set*; f(n) is a member of this set if and only if there exist constants c and N such that $0 \le f(n) \le c\ g(n)$, for all $n \ge N$

- We *should* write $f(n) \in O(g(n))$
- But by convention, we write $f(n) = O(g(n))$

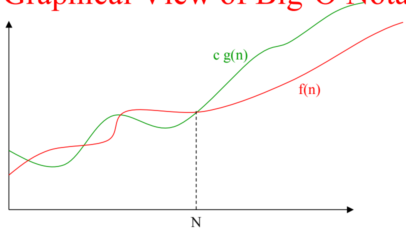
Claim: $n^2 + n = O(n^2)$

We know $n \le n^2$ for $n \ge 1$

So $n^2 + n \le 2\ n^2$ for $n \ge 1$

So by definition, $n^2 + n = O(n^2)$ for c=2 and N=1

A Graphical View of Big-O Notation



- To prove that f(n) = O(g(n)):
  - Find an N and c such that $0 \le f(n) \le c\ g(n)$, for all $n \ge N$
  - We call the pair (c, N) a *witness pair* for proving that f(n) = O(g(n))

Big-O Examples

Claim: 100 n + log n = O(n)

We know $\log n \le n$ for $n \ge 1$

So $100\ n + \log n \le 101\ n$ for $n \ge 1$

So by definition, 100 n + log n = O(n) for c=101 and N=1

Claim: $\log_B n = O(\log n)$

Let k = log n

Then $n = 2^k$ and (the subscripts are too messy; switch to board)

Question: Which grows faster: sqrt(n) or log n?

Simple Big-O Examples

- Let $f(n) = 3n^2 + 6n - 7$
  - Claim $f(n) = O(n^2)$
  - Claim $f(n) = O(n^3)$
  - Claim $f(n) = O(n^4)$
  - …
- g(n) = 4n log n + 34 n – 89
  - Claim g(n) = O(n log n)
  - Claim $g(n) = O(n^2)$
- $h(n) = 20 * 2^n + 40$
  - Claim $h(n) = O(2^n)$
- a(n) = 34
  - Claim a(n) = O(1)
- Only the *leading* term (the term that grows most rapidly) matters

Problem-Size Examples

- Suppose we have a computing device that can execute 1000 operations per second; how large a problem can we solve?

	1 second	1 minute	1 hour
n	1000	60,000	3,600,000
n log n	140	4893	200,000
n^2	31	244	1897
$3n^2$	18	144	1096
n^3	10	39	153
2^n	9	15	21

Commonly Seen Time Bounds

O(1)	constant	excellent
O(log n)	logarithmic	excellent
O(n)	linear	good
O(n log n)		pretty good
$O(n^2)$	quadratic	OK
$O(n^3)$	cubic	maybe OK
$O(2^n)$	exponential	too slow

Related Notations

- Big-Omega

Definition: f(n) is a member of the set Ω(g(n)) if there exists constants c and N such that $0 \le c\, g(n) \le f(n)$, for all $n \ge N$

- Big-Theta

Definition: f(n) is a member of the set Θ(g(n)) if f(n) = O(g(n)) and f(n) = Ω(g(n))

Worst-Case/Expected-Case Bounds

- We can't possibly determine time bounds for all possible inputs of size n
- Simplifying assumption #4: Determine number of steps for either
  - worst-case or
  - expected-case
- Worst-case
  - Determine how much time is needed for the *worst possible* input of size n
- Expected-case
  - Determine how much time is needed *on average* for all inputs of size n

Our Simplifying Assumptions

1. Use the *size* of the input rather than the input itself
2. Count the number of "*basic steps*" rather than computing exact times
3. Multiplicative constants aren't important (i.e., use big-O notation)
4. Determine number of steps for either
   - worst-case or
   - expected-case

Worst-Case Analysis of Searching

- Linear Search (check each element)

```
static boolean find (int[ ] a, int item) {
    for (int i = 0; i < a.length; i++) {
            if (a[i] == item) return true;
            }
    return false;
    }
```

For Linear Search, worst-case time is O(n)

For Binary Search, worst-case time is O(log n)

- Binary Search

```
static boolean find (int[ ] a, int item) {
    int low = 0;
    int high = a.length - 1;
    while (low <= high) {
            int mid = (low+high)/2;
            if (a[mid] < item)
                    low = mid+1;
            else if (item < a[mid])
                    high = mid - 1;
            else return true;
            }
    return false;
    }
```

Analysis of Matrix Multiplication

Code for multiplying n-by-n matrices A and B:

```
for (i = 0; i<n; i++)
    for (j = 0; j < n; j++)
        for (k = 0; k < n; k++)
            C[i][j] = C[i][j] + A[i][k] * B[k][j];
```

- By convention, matrix problems are measured in terms of n, the number of rows and columns
  - Note that the input size is really $2n^2$, not n
  - Worst-case time is $O(n^3)$
  - Expected-case time is also $O(n^3)$

Remarks

- Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity
  - For example, you can usually ignore everything that is not in the innermost loop. Why?
- Main difficulty:
  - Determining runtime for recursive programs

Summary

- Asymptotic complexity
  - Used to measure of time (or space) required by an algorithm
  - Measure of the *algorithm*, not the *problem*
- Searching array
  - Linear search: O(n) worst-case time
  - Binary search: O(log n) worst-case time
- Matrix operations:
  - Note: n = number-of-rows = number-of-columns
  - Matrix-vector product: $O(n^2)$ worst-case time
  - Matrix-matrix multiplication: $O(n^3)$ worst-case time

Why Bother with Runtime Analysis?

- Computers are so fast these days that we can do whatever we want using just simple algorithms and data structures, can't we?
- Well…not really; data-structure/algorithm improvements can be a *very big* win
- Scenario:
  - A runs in n^2 msec
  - A' runs in $n^2/10$ msec
  - B runs in 10 n log n msec

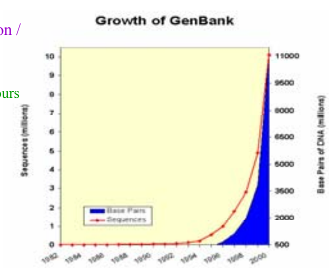
- Problem of size $n=10^3$
  - A: 10^3 sec ≈ 17 minutes
  - A': 10^2 sec ≈ 1.7 minutes
  - B: 10^2 sec ≈ 1.7 minutes
- Problem of size $n=10^6$
  - A: 10^9 sec ≈ 30 years
  - A': 10^8 sec ≈ 3 years
  - B: 2×10^5 sec ≈ 2 days

1 day = 86,400 sec ≈ 10^5 sec

1,000 days ≈ 3 years

Algorithms for the Human Genome

- Human genome = 3.5 billion nucleotides ~ 1 Gb
  - @ 1 base-pair instruction / μsec
    - $n^2 \Rightarrow$ 388445 years
    - n log n ⇒ 30.824 hours
    - n ⇒ 1 hour



Limitations of Runtime Analysis

- Big-O can hide a large constant
  - Example: Selection
  - Example: small problems
- The specific problem you want to solve may not be the worst case
  - Example: Simplex method for linear programming
- Your program may not be run often enough to make analysis worthwhile
  - Example: one-shot vs. every day
- You may be analyzing and improving the wrong part of the program
  - *Very* common situation
  - Should use *profiling* tools