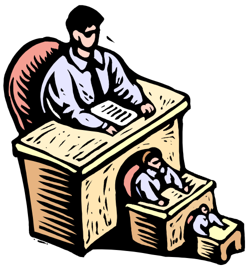
Recursion



Lecture 4
CS211 – Fall 2005

Announcements

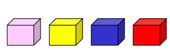
- Assignment 2 is online (since Friday)
  - Due date: Wednesday, September 14
  - Recommendation: Start now
- If you would like a partner for A2
  - Sign up sheet
  - Name and netID
- Be sure to "form your group" on CMS!
  - It does *not* happen automatically
- For extra Java help
  - Lots of consulting/office-hours are available
    - General Java-help is more easily available in week *before* assignment is due
  - Can set up individual meetings with TAs via email

Recursion

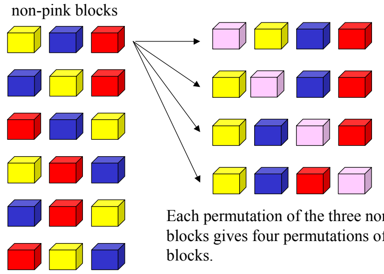
- Recursion is a powerful technique for specifying functions, sets, and programs
- Recursively-defined functions and programs
  - factorial
  - combinations
  - differentiation of polynomials
- Recursively-defined sets
  - grammars
  - expressions
  - data structures (lists, trees, ...)

The Factorial Function (n!)

- Define $n! = n\cdot(n-1)\cdot(n-2)\cdots3\cdot2\cdot1$ read: "n factorial"
- E.g., $3! = 3\cdot2\cdot1 = 6$
- By convention, $0! = 1$
- The function int $\rightarrow$ int that gives n! on input n is called the factorial function.
- n! is the number of permutations of n distinct objects
  - There is just one permutation of one object. $1! = 1$
  - There are two permutations of two objects: $2! = 2$
    1 2 2 1
  - There are six permutations of three objects: $3! = 6$
    1 2 3 1 3 2 2 1 3 2 3 1 3 1 2 3 2 1
- If $n > 0$, $n! = n\cdot(n-1)!$

Permutations of 

Permutations of non-pink blocks



Each permutation of the three non-pink blocks gives four permutations of the four blocks.

Total number $= 4\cdot6 = 24 = 4!$

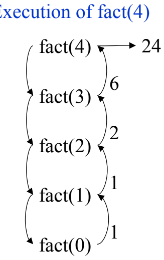
A Recursive Program

$0! = 1$

$n! = n\cdot(n-1)!, \ n > 0$

```
static int fact(int n) {
   if (n == 0) return 1;
   else return n*fact(n-1);
}
```

Execution of fact(4)



General Approach to Writing Recursive Functions

1. Try to find a parameter, say *n*, such that the solution for *n* can be obtained by combining solutions to the *same* problem with smaller values of *n* (e.g., chess-board tiling, factorial)
2. Figure out the base case(s) — small values of *n* for which you can just write down the solution (e.g., 0! = 1)
3. Verify that for any value of *n* of interest, applying the reduction of step 1 repeatedly will ultimately hit one of the base cases

The Fibonacci Function

- Mathematical definition:
  - $fib(0) = 0$
  - $fib(1) = 1$ two base cases!
  - $fib(n) = fib(n-1) + fib(n-2), \; n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, …

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

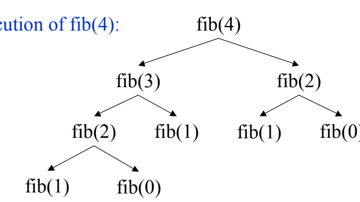
Fibonacci (Leonardo Pisano, 1170–1240?)

Statue in Pisa, Italy
Giovanni Paganucci, 1863

Recursive Execution

```
static int fib(int n) {
   if (n == 0) return 0;
   else if (n == 1) return 1;
   else return fib(n-1) + fib(n-2);
}
```

Execution of fib(4):



Combinations (a.k.a. Binomial Coefficients)

How many ways can you choose r items from a set S of n distinct elements? $\binom{n}{r}$ "n choose r"

$\binom{5}{2}$ = number of 2-element subsets of S = {A,B,C,D,E}

- 2-element subsets containing A: $\binom{4}{1}$
  {A,B}, {A,C}, {A,D},{A,E}
- 2-element subsets not containing A: $\binom{4}{2}$
  {B,C},{B,D},{B,E},{C,D},{C,E},{D,E}

Therefore, $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

- You can also show that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

Pascal's triangle

$$= \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Combinations

These are also called binomial coefficients because they appear as coefficients in the expansion of the binomial power $(x + y)^n$:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$
$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

Combinations have two base cases

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

Two base cases

- Coming up with right base cases can be tricky!
- General idea:
  - Determine argument values for which recursive case does not apply
  - Introduce a base case for each one of these
- Rule of thumb: (not always valid) if you have *r* recursive calls on right hand side, you may need *r* base cases.

Recursive Program for Combinations

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \quad n > r > 0$$
$$\binom{n}{n} = 1$$
$$\binom{n}{0} = 1$$

```
static int combs(int n, int r){     //assume n>=r>=0
   if (r == 0 || r == n) return 1; //base cases
   else return combs(n-1,r) + combs(n-1,r-1);
}
```

Polynomial Differentiation

Inductive cases:
d(uv)/dx = u dv/dx + v du/dx
d(u+v)/dx = du/dx + dv/dx
Base cases:
dx/dx = 1
dc/dx = 0

Example:

d(3x)/dx = 3dx/dx + x d(3)/dx = 3·1 + x·0 = 3

Positive Integer Powers

$a^n = a \cdot a \cdot a \cdots a$ (n times)

Alternative description:

$a^0 = 1$
$a^{n+1} = a \cdot a^n$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   else return a*power(a,n-1);
}
```

A Smarter Version

- Power computation:
  - $a^0 = 1$
  - If n is nonzero and even, $a^n = (a^{n/2})^2$
  - If n is odd, $a^n = a \cdot (a^{n/2})^2$
    - Java note: If x and y are integers, "x/y" returns the integer part of the quotient
- Example:
  $a^5 = a \cdot (a^{5/2})^2 = a \cdot (a^2)^2 = a \cdot ((a^{2/2})^2)^2 = a \cdot (a^2)^2$
  Note: this requires 3 multiplications rather than 5!

- What if n were higher?
  - savings would be higher
- This is much faster than the straightforward computation
  - Straightforward computation: n multiplications
  - Smarter computation: log(n) multiplications

Smarter Version in Java

- $n = 0$: $a^0 = 1$
- n nonzero and even: $a^n = (a^{n/2})^2$
- n odd: $a^n = a \cdot (a^{n/2})^2$

```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

Implementation of Recursive Methods

local variable

parameters

```
static int power(int a, int n) {
   if (n == 0) return 1;
   int halfPower = power(a,n/2);
   if (n%2 == 0) return halfPower*halfPower;
   return halfPower*halfPower*a;
}
```

- The method has two parameters and a local variable
- Why aren't these overwritten on recursive calls?

Implementation of Recursive Methods

- Key idea:
  - Use a stack to remember parameters and local variables across recursive calls
  - Each method invocation gets its own stack frame
- A stack frame contains storage for
  - Local variables of method
  - Parameters of method
  - Return info (return address and return value)
  - Perhaps other bookkeeping info

Stacks

stack grows

top element
2nd element
3rd element
…
…
bottom element

top-of-stack pointer

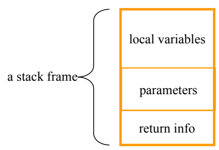
- Like a stack of plates
- You can push data on top or pop data off the top in a LIFO (last-in-first-out) fashion
- A queue is similar, except it is FIFO (first-in-first-out)

java.lang.Stack

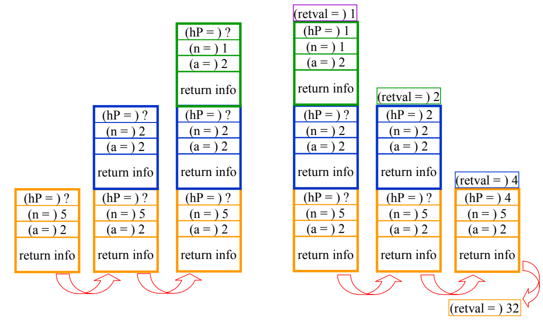
Stack()	Creates an empty Stack
boolean empty()	Tests if the stack is empty
E peek()	Looks at the object at the top of the stack without removing it from the stack
E pop()	Removes the object at the top of the stack and returns that object as the value of the function
push(E item)	Pushes an item onto the top of the stack
int search(E o)	Returns the position of the given item on the stack

Stack Frame

- A new stack frame is pushed with each recursive call
- The stack frame is popped when the method returns
  - Leaving a return value (if there is one) on top of the Stack



Example: power(2, 5)



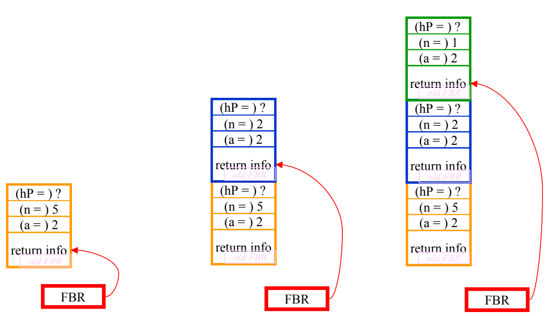
How Do We Keep Track?

- At any point in execution, many invocations of *power* may be in existence
  - Many stack frames (all for *power*) may be in Stack
  - Thus there may be several different versions of the variables *a* and *n*
- How does processor know which location is relevant at a given point in the computation?

Answer: Frame Base Register

- Computational activity takes place only in the topmost (most recently pushed) stack frame
  - Special register called Frame Base Register (FBR) keeps track of where the topmost frame is
- Using the FBR
  - When a method is invoked, a frame is created for that method invocation, and FBR is set to point to that frame
  - When the invocation returns, FBR is restored to what it was before the invocation
- How does machine know what value to restore in FBR?
  - This is part of the return info in the stack frame

FBR



Conclusion

- Recursion is a convenient and powerful way to define functions
- Problems that seem insurmountable can often be solved in a "divide-and-conquer" fashion:
  - Reduce a big problem to smaller problems of the same kind, solve the smaller problems
  - Recombine the solutions to smaller problems to form solution for big problem
- Important application (next lecture): parsing of languages