Priority Queues
& Heaps

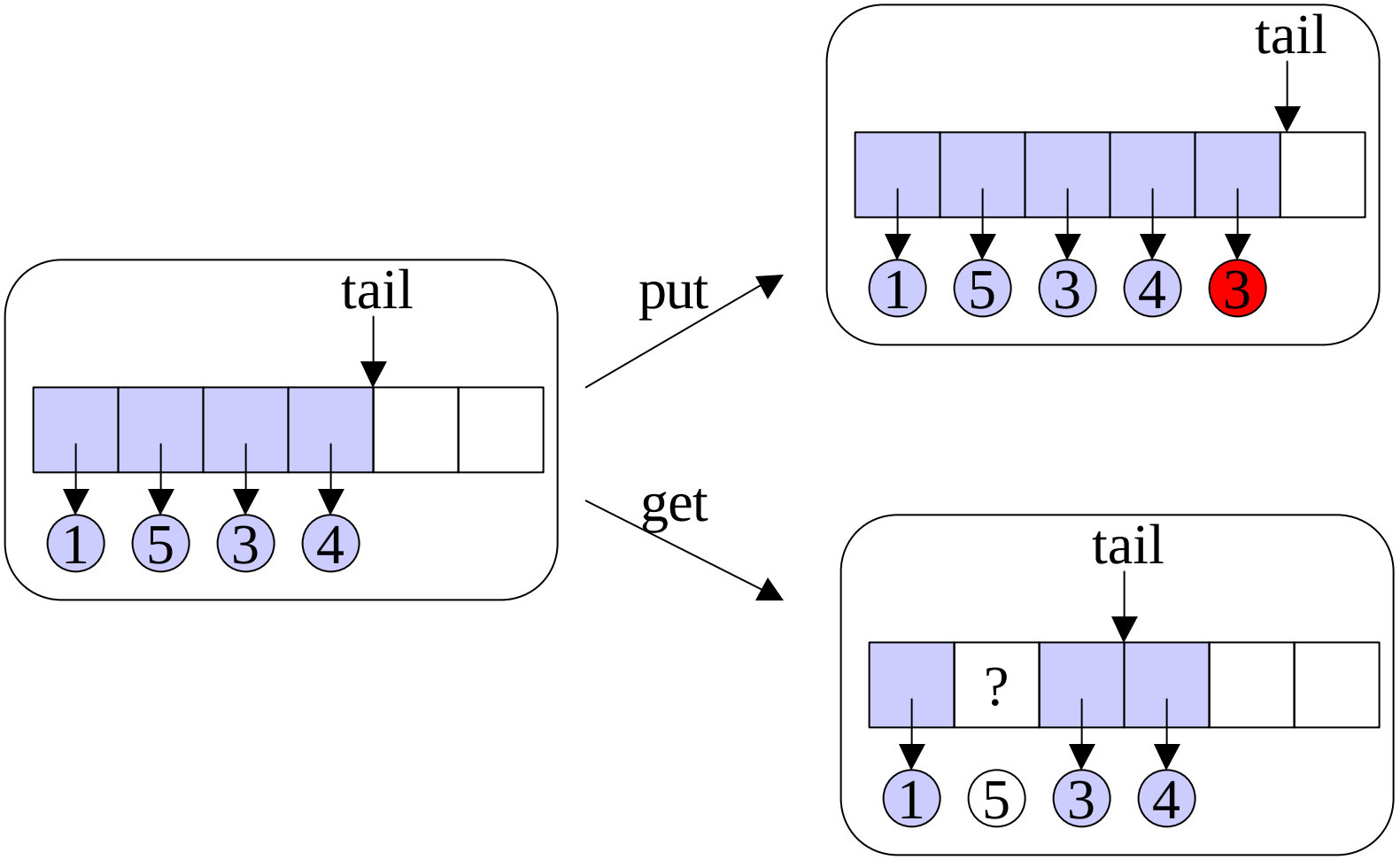
Operations

- put: insert item with a priority
- get: remove the item with highest priority, and break ties with FIFO ordering

- For simplicity:
  - we use an int as the object *and* the priority
  - In practice: need <int priority, Object elt> pair

Niave Solution: Prio-Q As Array

- put: add item to end of array
- get: search for highest priority item

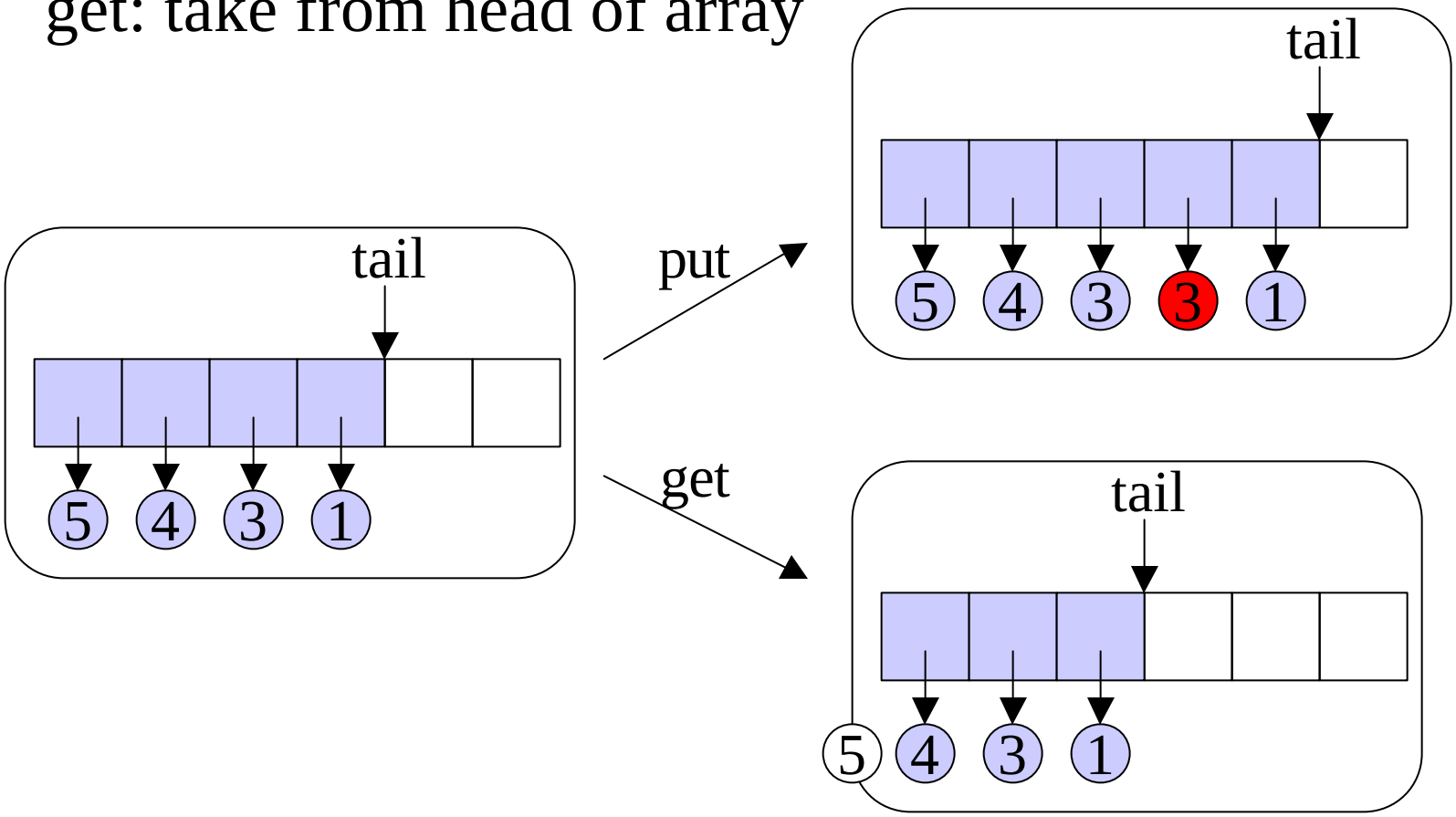


Prio-Q as Array

- put: O(1) assuming no array doubling
- get: O(n)
  - O(n) to find element, O(1) to patch hole

Prio-Q as Sorted Array

- Maintain array in sorted order, largest first
- put: add item to “correct place” in array
- get: take from head of array

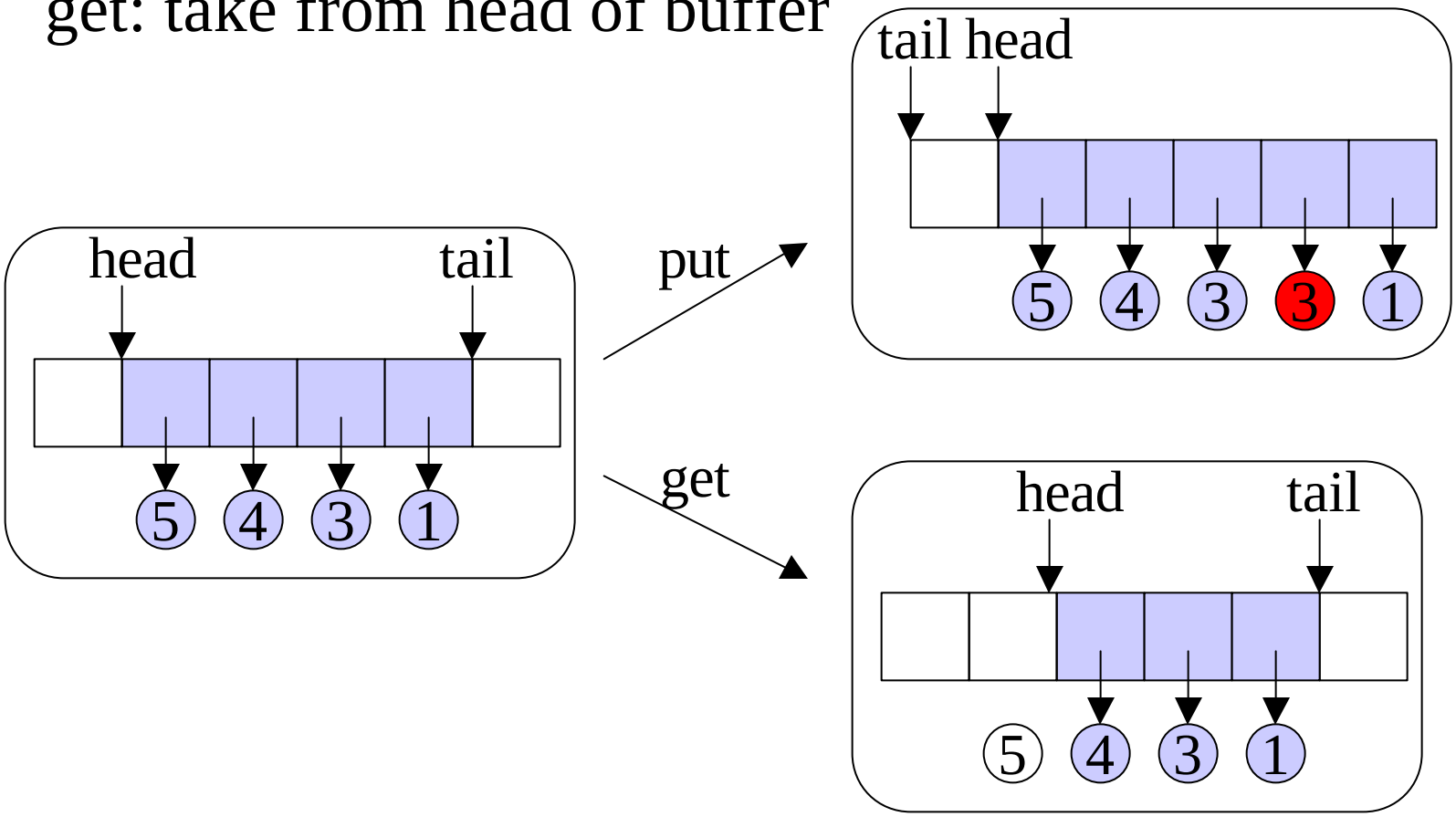


Prio-Q as Sorted Array

- put: O(n)
  - O(n) to find correct position + O(n) to shift remaining elements out of the way
  - Can we use binary search?
- get: O(n)
  - Need to slide all elements to the left

Prio-Q as Sorted Circular Buffer

- Maintain circular buffer in sorted order, largest first
- put: add item to “correct place” in buffer
- get: take from head of buffer



Prio-Q as Sorted Circular Buffer

- put: O(n)
  - O(n) to find correct position + O(n) to shift remaining elements out of the way
  - Can we use binary search?
- get: O(1)

Prio-Q as Sorted Linked List

- put: O(n)
  - O(n) to find item (linear search) + O(1) to splice it out
- get: O(1)

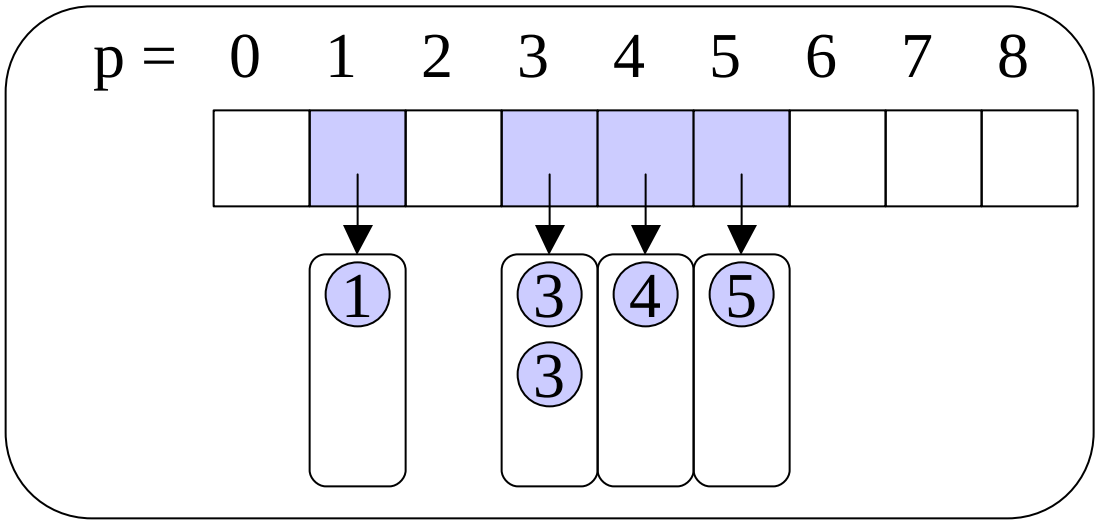
Special Case: at most *p* priorities

- Many applications only use a fixed number of priorities:
  - 8-Puzzle with # tiles correct as heuristic: *p* in 0..8
  - Java task priorities: usually 1..10
  - Email priority: very low, low, normal, high, urgent
- Cool implementation:
  - Use array of *p* (FIFO) queues, one for each priority value

put: O(1)

get: O(P)

P = # priorities

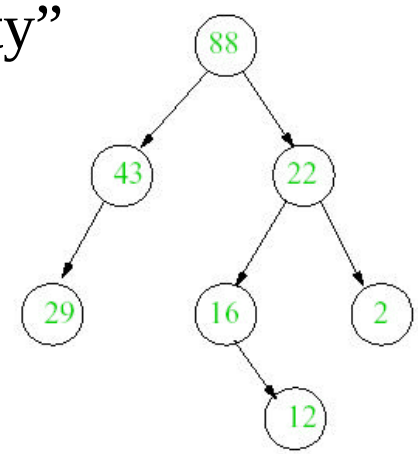


General Solution: Heap

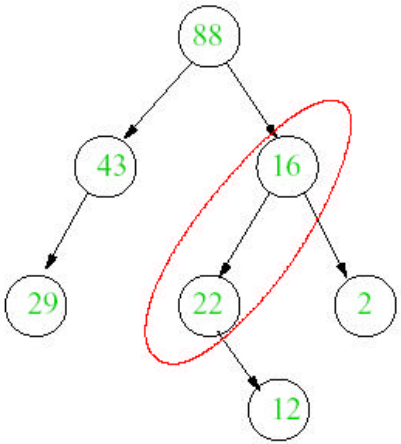
A *heap* is a tree in which

1. an integer is stored at each node
2. integer stored at node is >= integer stored at any of its children

"the heap property"



Heap



Not a heap

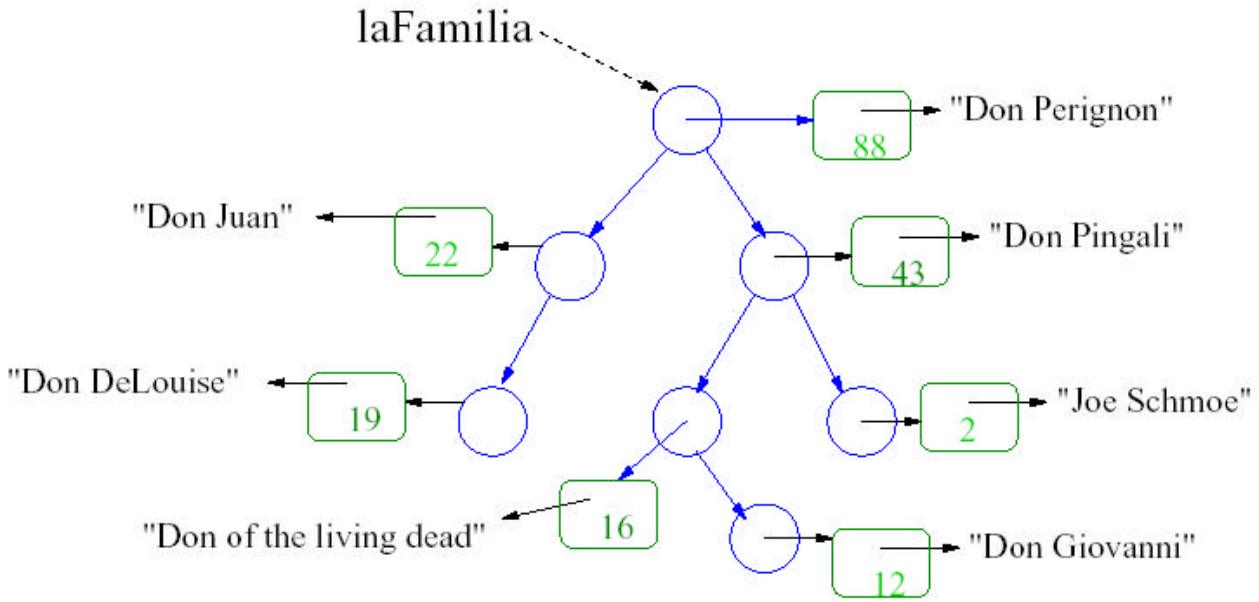
For now, we only care about *binary* trees

Heaps in Reality

- Ages of people in a family tree
  - Parent older than children
  - But your uncle may be younger than you
- Salaries
  - boss makes more than underlings
  - but temp worker in one division can make more than a supervisor in a different division

Running Example: Crime Family

- Nodes contain name and “ruthlessness” (= # crimes committed, an integer)
- Boss is always more ruthless than a subordinate



Heap of Priority Queue Elements

Prio-Q as Heap

- get: must return element with largest priority

- put: insert somewhere in the heap and maintain/restore the heap property
  - “heapify” = restore the heap property

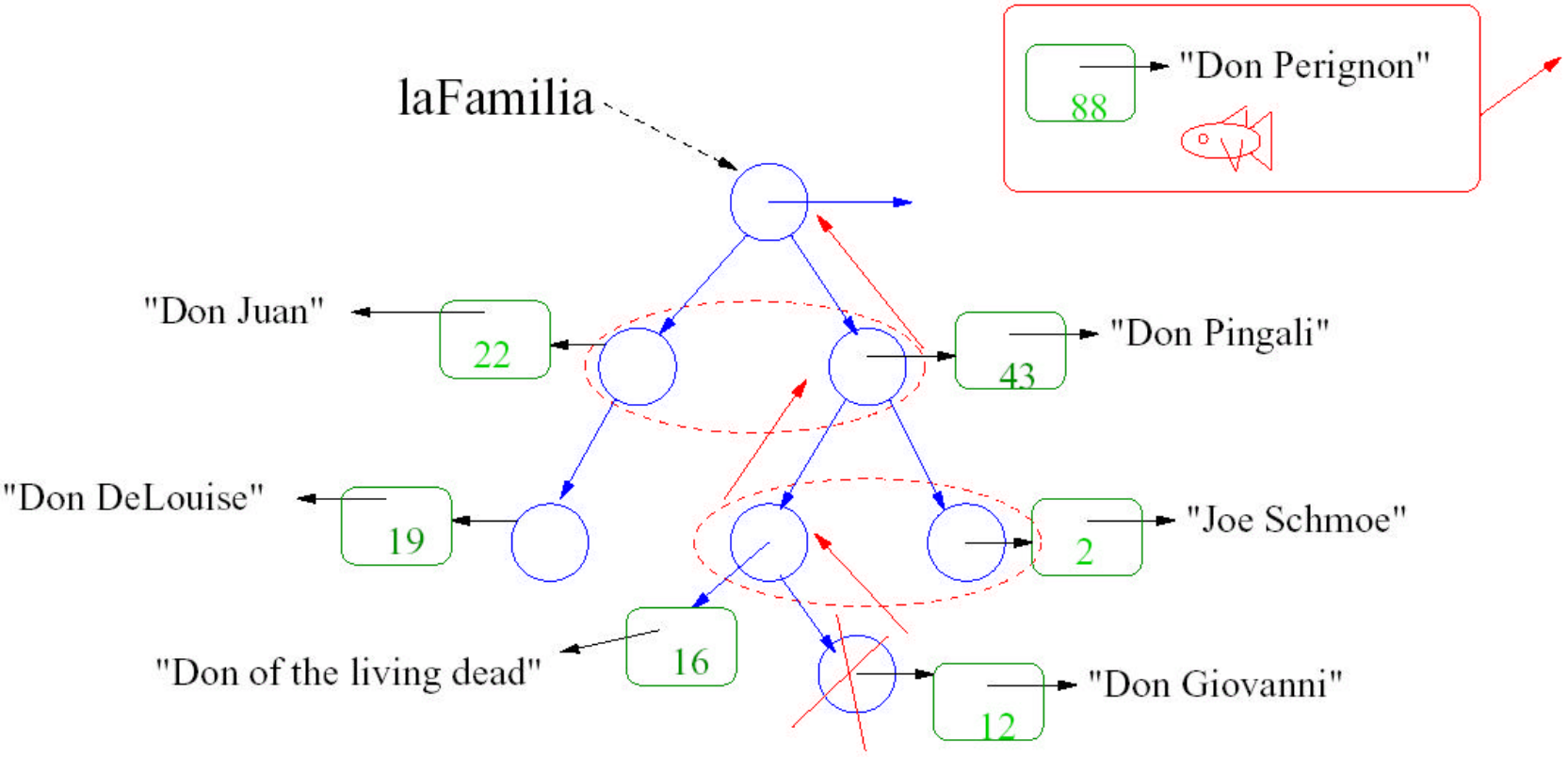
get

- Highest priority always at root
- Remove root, then fill hole

Solution #1:

  - promote the highest-priority child
  - recurse
  - until you promote a leaf element
  - delete the now-empty leaf

Heapify #1: Restore the heap property

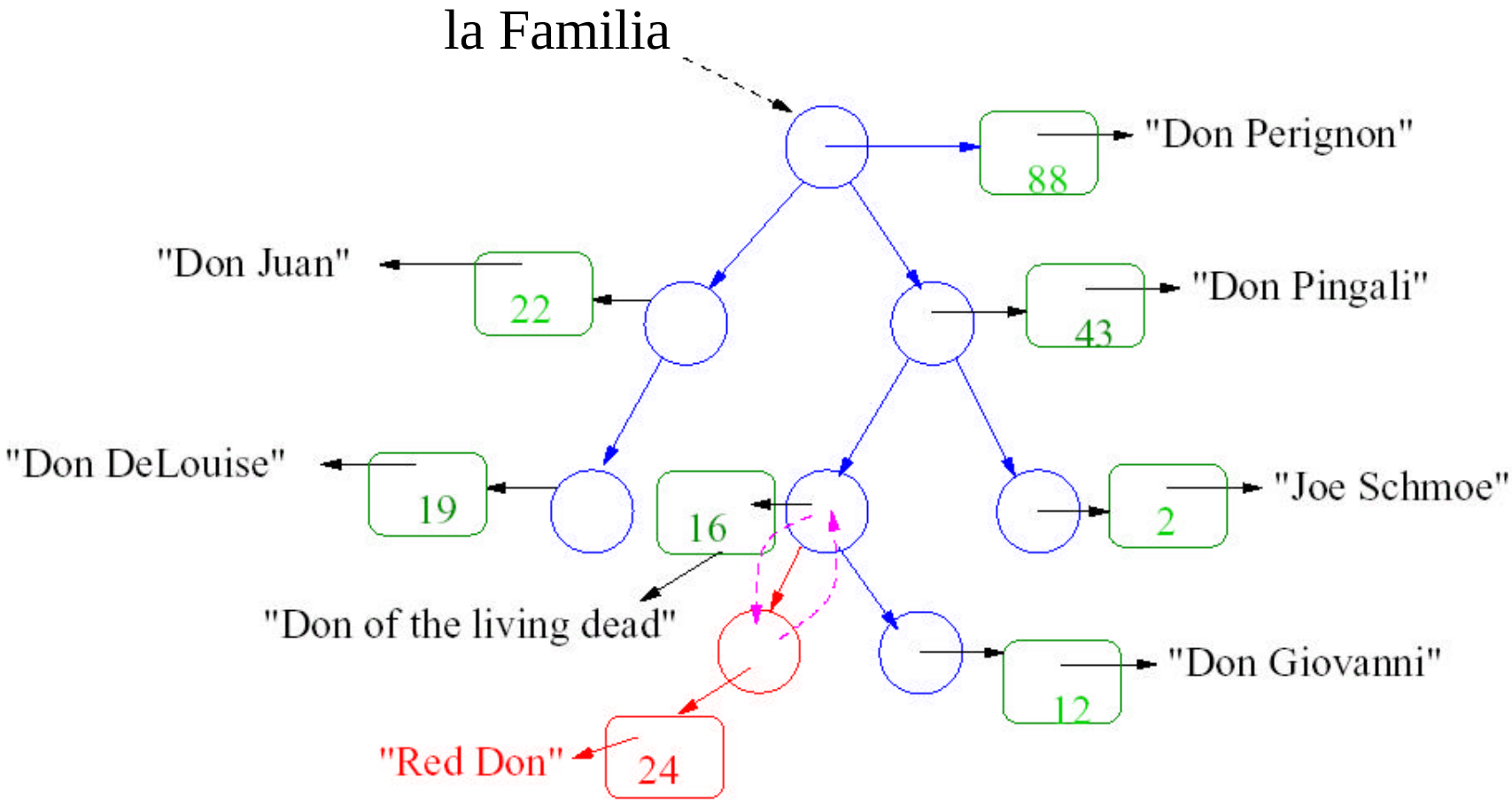


Heapifying after removing root element

put

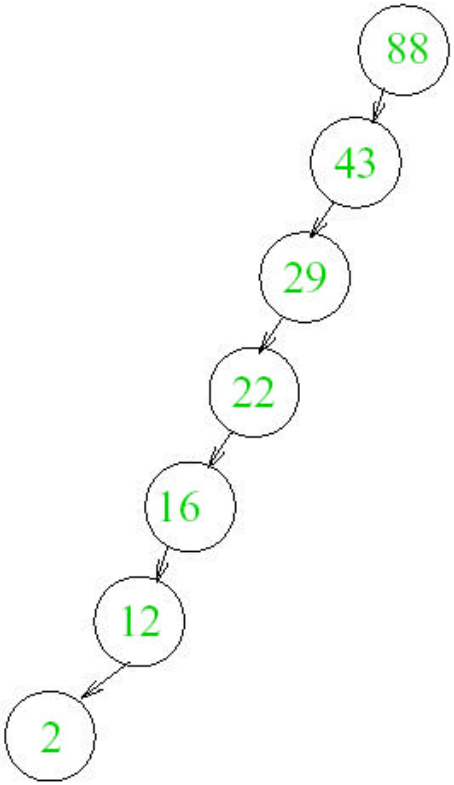
- Insert new element into a new leaf node, anywhere in the tree
- Then “bubble up” the new element until heap property is restored
  - compare new element against parent
  - if needed, swap then repeat in the new position
  - worst case: bubble up all the way to the root

Bubble up

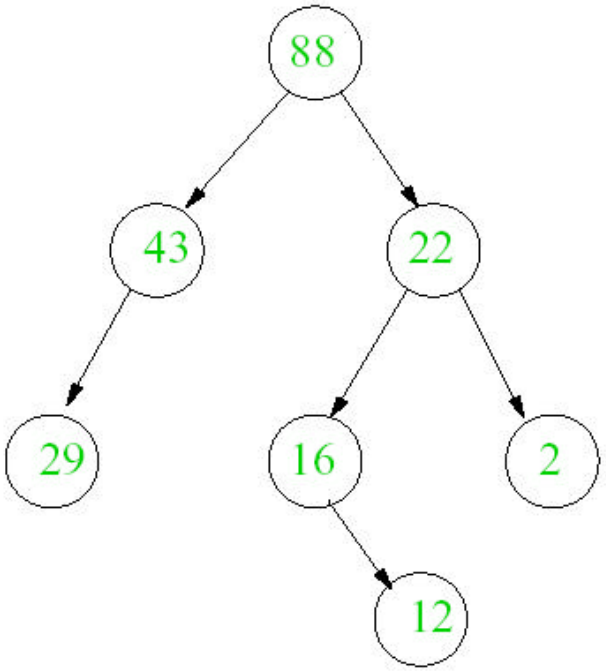

la Familia
"Don Perignon"
88
"Don Juan"
22
"Don Pingali"
43
"Don DeLouise"
19
16
2
"Joe Schmoe"
"Don of the living dead"
"Don Giovanni"
12
"Red Don"
24

Prio-Q as Heap

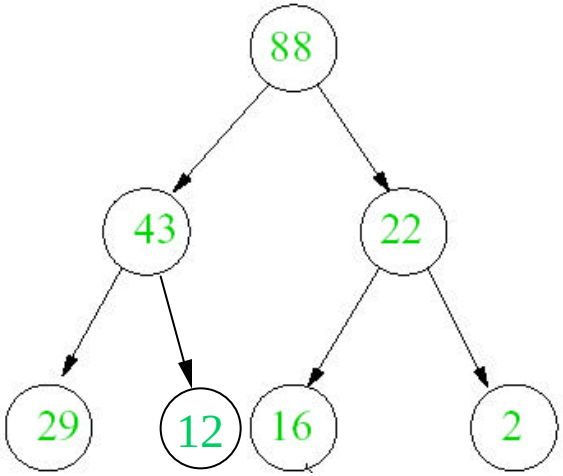
- put: O(n) worst case, but O(log n) if tree is nice
- get: O(n) worst case, but O(log n) if tree is nice



Not Nice



Nice

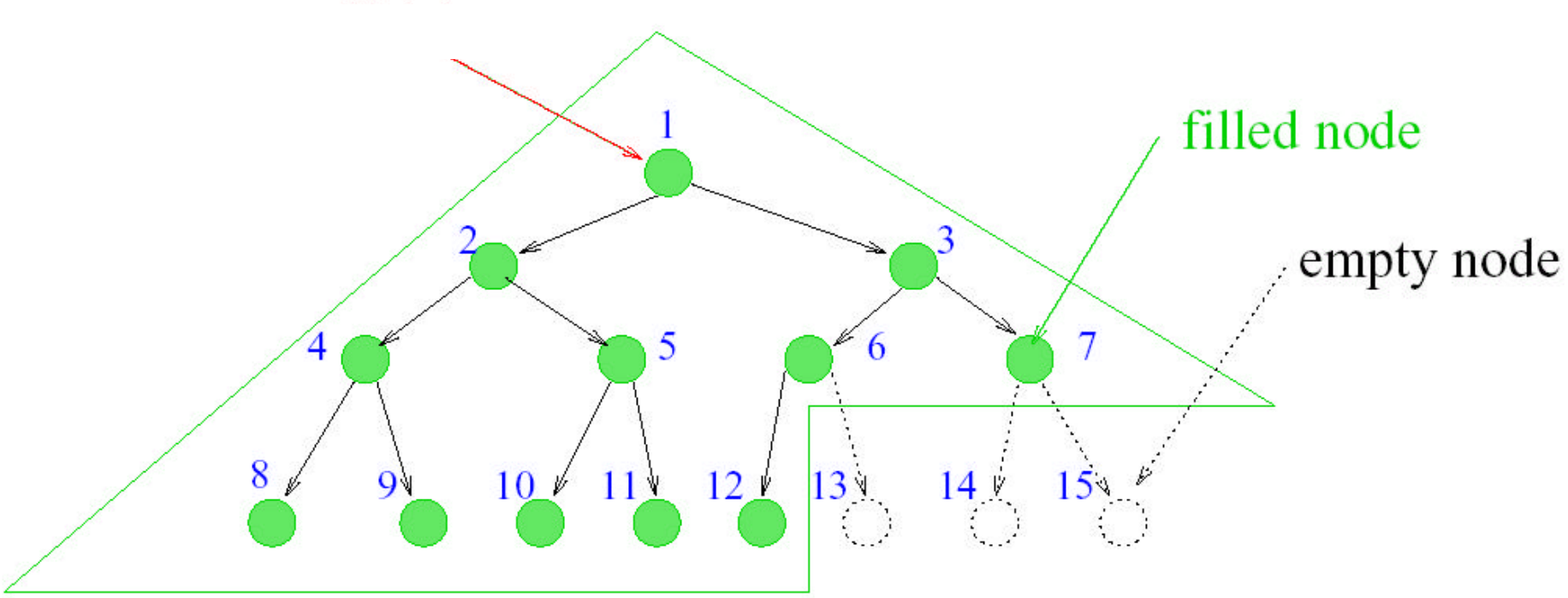


Very Nice

Keeping the Tree Fat & Short

A *complete binary tree* is a tree in which:

1. Nodes are numbered in a bfs fashion (the “position”)
2. Node at position *n* is filled only if all nodes in positions less than *n* are filled

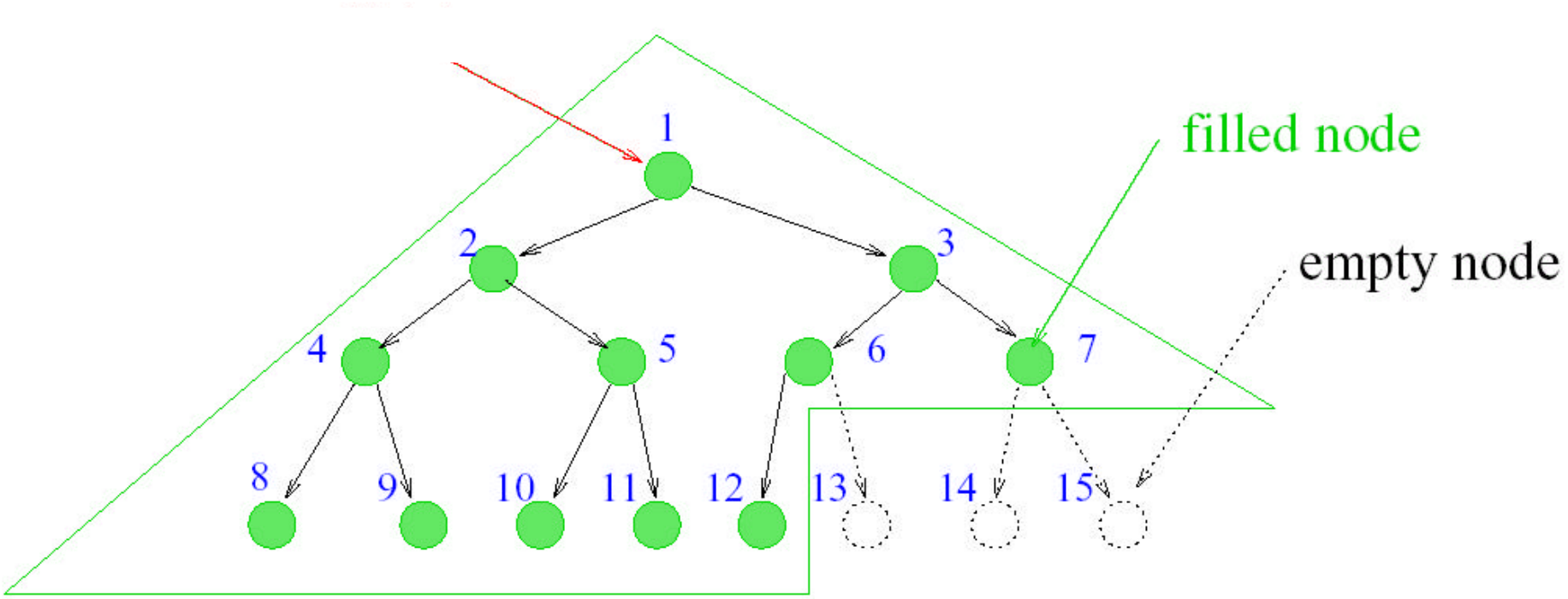


Heap As Complete Binary Tree

- Will keep tree short and fat, and us happy
- Get: will ruin the complete binary tree property, since the bottom-right-most node might not be the one to get promoted
- Put: need to create new node in the first empty position
  - easy to name (next empty position = current size + 1)
  - how to find the position once we have the number?

Quick Diversion: Binary Numbers

- Finding empty position 13 given root (position 1)
- $13 = 8 + 4 + 0 + 1 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$
  $= 1101$ (binary)

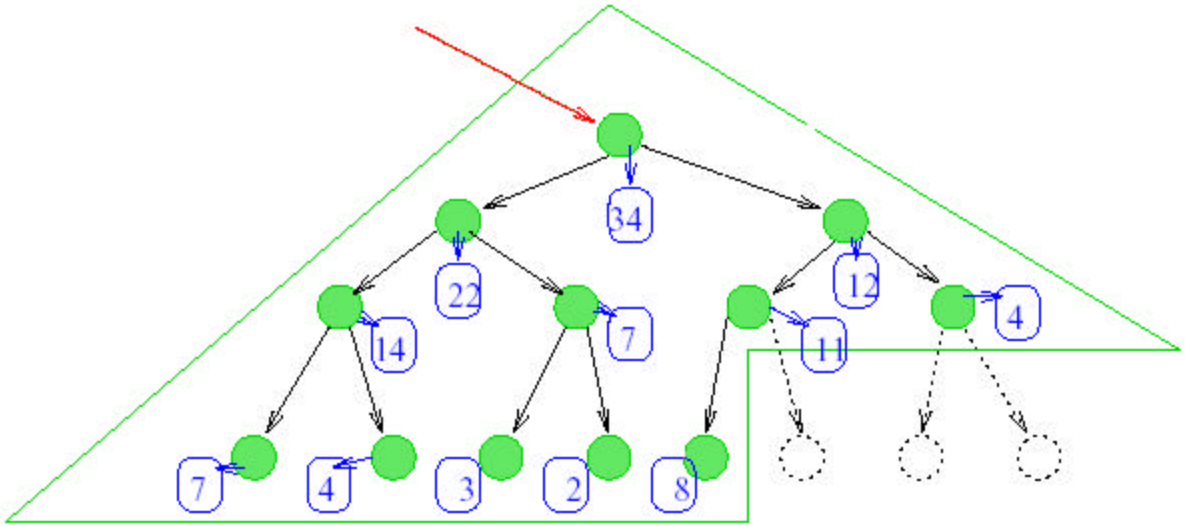


So we can find any position in log n steps

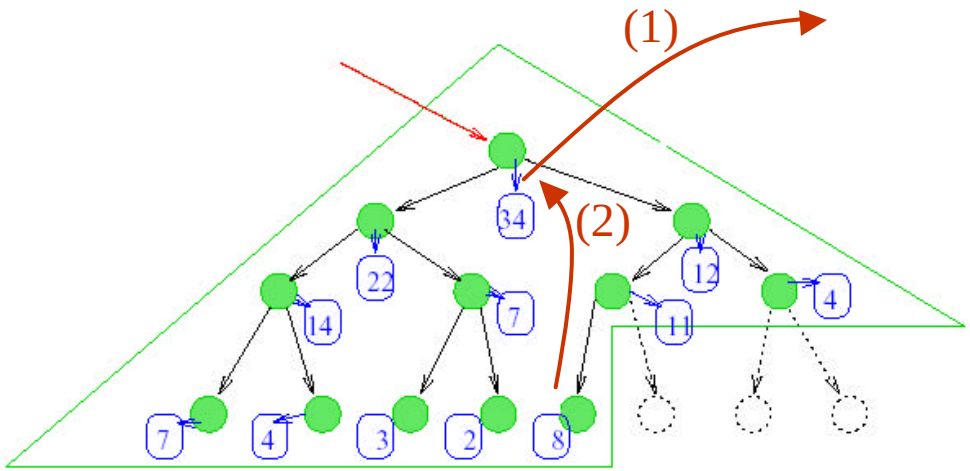
put

- If current size = n, insert at position n’ = n+1
  - Convert n’ to binary
  - substitute R for 1, and L for 0
  - follow links to the position
- Insert new element
- Bubble up, as before

get is still broken



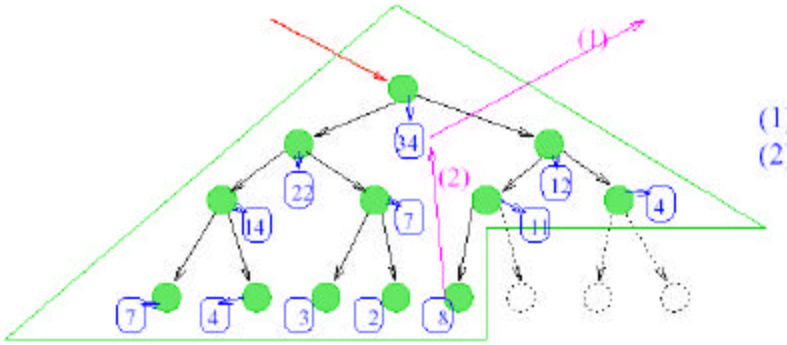
Fixing get



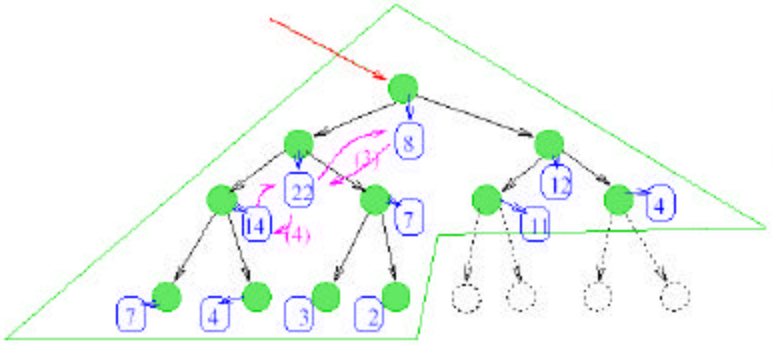
steps:

1. Extract root
2. Promote “last” element directly into root

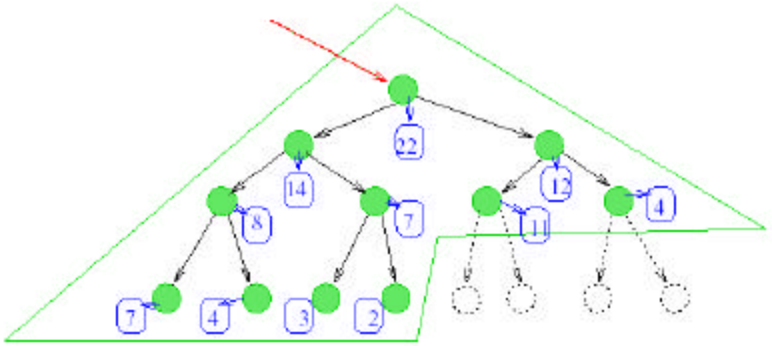
Fixing get



(1) extract root
(2) promote "last" element into root



(3) exchange 8 and 22
(4) exchange 8 and 14



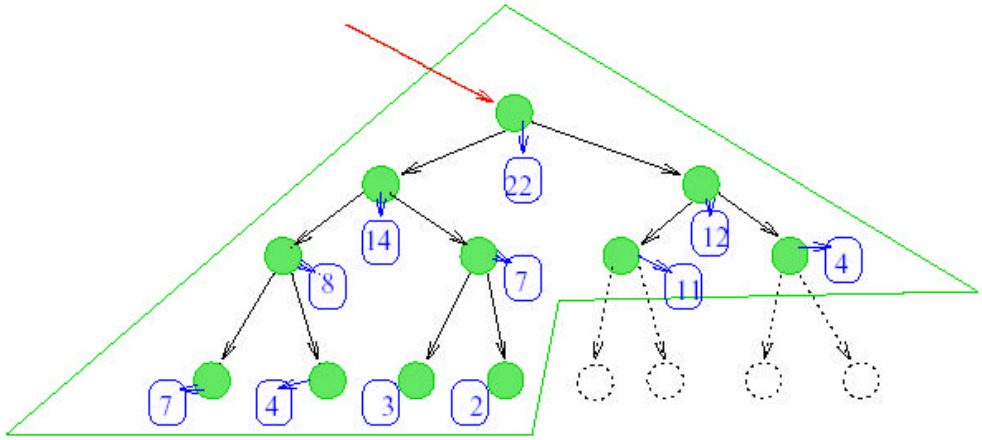
final heap: complete tree once again

Prio-Q as Heap

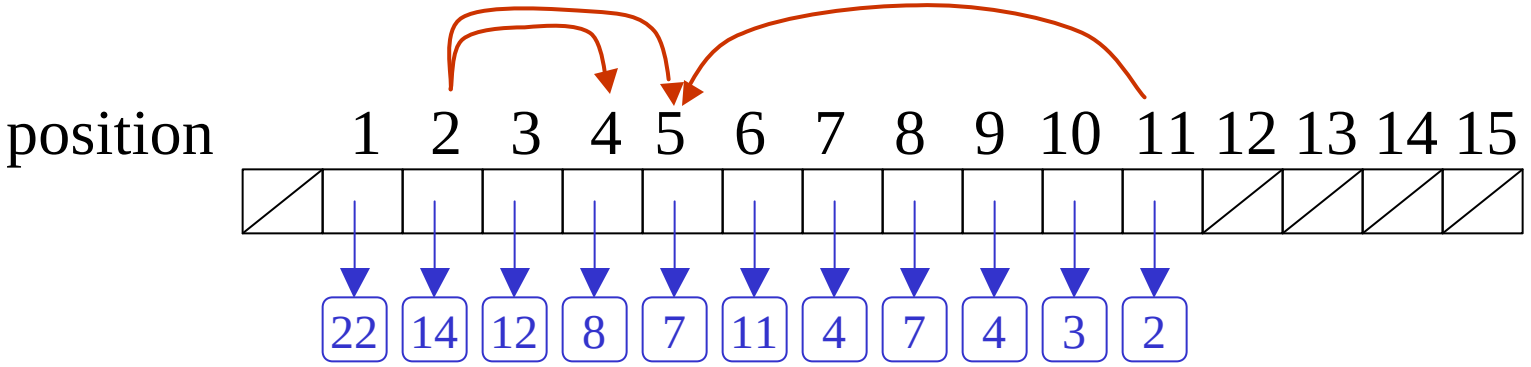
- Use a heap that is a complete binary tree
  - Keep track of size
- get: O(log n)
  - remove root, promote last leaf node, bubble down
- put: O(log n)
  - insert into “last+1” leaf node, bubble up

Neat Trick for complete binary trees

- Don’t store as Tree Cells
- Instead, store in an array, indexed by position #



children(i) = 2i and 2i+1 parent(i) = i/2



Heap Construction

- build-heap: Given *n* elements in an array, rearrange to build a heap
- Obvious solution:
  - Start with empty array
  - Do heap insert for each element
  - 1 + n * log n → O(n log n)
- There is a faster way…