Induction Exercises

These exercises are for your own benefit, and are entirely optional. Feel free to collaborate and share your answers with other students. These problems are taken from various sources at Cornell and on the Internet, too numerous to cite individually. **Some hints are at the bottom of the page.**

1. Find a formula for summing the cubes of integers, and prove your formula correct. That is, find and prove a closed form for *third(n)* = $1^3 + 2^3 + ... + n^3$. MATLAB can be helpful in solving systems of equations (type "help solve" and "help splash" for help in MATLAB). So can Mathematica or Maple.

2. Using induction, show that $4^n + 15n - 1$ is divisible by 9 for all $n >= 1$.

3. What is wrong with the following proof that all horses have the same color?

> Let P(n) be the proposition that all the horses in a set of n horses are the same color. Base case: Clearly, P(1) is true. Now assume that P(n) is true. That is, assume that all the horses in any set of n horses are the same color. Consider any $n + 1$ horses; number these as horses 1, 2, 3,..., n, $n + 1$. Now the first n of these horses all must have the same color, and the last n of these must also have the same color. Since the set of the first n horses and the set of the last n horses overlap, all $n + 1$ must be the same color. This shows that P($n + 1$) is true and finishes the proof by induction.

4. Prove *Bernoulli's Inequality*: $1 + nh <= (1+h)^n$ for $n >= 0$, and where $h > -1$.

5. Prove for all $n >= 0$, that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$. Note: $2 \cdot 2!$ means 2*(2!).

6. Prove that $n^2 - 1$ is divisible by 8 for all odd positive integers n.

7. We already have seen that $2^n < n!$ for $n >= 4$. Now show that $n! < n^n$ for $n > 1$.

8. The Fibonacci numbers have a number of cute properties. Prove the following:
   - (a) F_{4n} is divisible by 3 for every $n >= 1$.
   - (b) $1 < (F_{n+1} / F_n) < 2$ for all $n > 2$
   - (c) $(F_n)^2 = F_{n-1} F_{n+1} + (-1)^{n+1}$

9. Let D_n denote the number of ways to cover the squares of a 2-by-n board using plain dominos. Then it is easy to see that $D_1 = 1$, $D_2 = 2$, and $D_3 = 3$. Compute a few more values of D_n and guess an expression for the value of D_n and use induction to prove you are right. [Strong Induction]

10. This is an example of where "guess-and-check" can mislead you into thinking something is true. It appears that $n^2 + n + 41$ is prime for every $n >= 1$. Find a counter example. [Not by induction.]. Some history trivia: Fermat conjectured that all numbers of

the form $2^n + 1$ are prime if n is a power of 2, and he verified base cases for $n = 1, 2, 4, 8$, and 16 before the calculations got too tedious to do by hand. It took more than 100 years for Euler to notice that for $n = 32$ the conjecture fails: $2^{32} + 1 = 4294967297$ is divisible by 641. Of course, this was before computers – now we can generate primes with lots of digits (11713 digits, last I checked)!

Hints & Spoilers

3. Examine the first few values for n carefully. E.g., look at P(1), P(2), …

6. Rewrite using $n = 2m+1$, then do induction on m.

8. Prove a stronger property instead: F_{4n} is divisible by 3, and so are $1+F_{4n+1}$, $1+F_{4n+2}$, and $2+F_{4n+3}$.