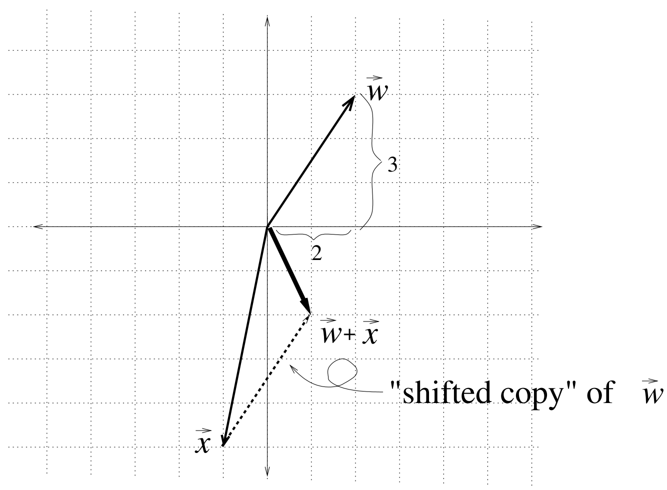
Computation, Information, and Intelligence (ENGRI/CS/INFO/COGST 172), Spring 2007
2/8/07: **Vector-operations reference sheet**

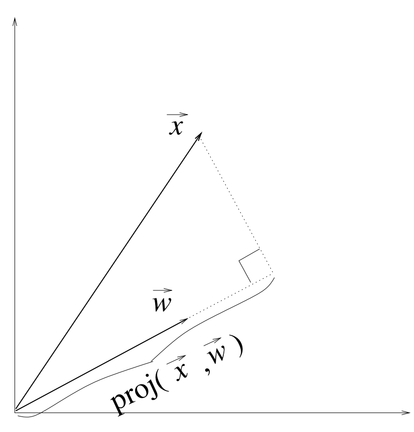
Vector addition For $\overrightarrow{w} = (w_1, w_2, \ldots, w_n)$ and $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)$, it's the case that

$$\overrightarrow{w} + \overrightarrow{x} \stackrel{\text{def}}{=} (w_1 + x_1, w_2 + x_2, \ldots, w_n + x_n);$$

that is, the operation is "component-wise". For example, $(2, 3) + (-1, -5) = (1, -2)$, as shown here:



An interpretation of the inner product Here is a schematic showing the *projection* $\text{proj}(\overrightarrow{x}, \overrightarrow{w})$ of $\overrightarrow{x}$ onto $\overrightarrow{w}$, which is a *signed* length: it is negative if the cosine of the angle between $\overrightarrow{x}$ and $\overrightarrow{w}$ is negative, that is, if $\overrightarrow{x}$ and $\overrightarrow{w}$ are more than $90°$ apart.[1]



The inner product relates to the vector projections in the following way.

$$\begin{aligned} \overrightarrow{w} \cdot \overrightarrow{x} &= \text{length}(\overrightarrow{w})\,\text{length}(\overrightarrow{x})\cos(\angle(\overrightarrow{w}, \overrightarrow{x})) \\ &= \text{length}(\overrightarrow{w})\,\text{proj}(\overrightarrow{x}, \overrightarrow{w}) \end{aligned}$$

because "cosine = adjacent over hypotenuse".

[1] We are abusing the conventional definition of the term projection for notational convenience.

Vector length The quantity $\text{length}(\vec{v})$ for a vector $\vec{v}$ can be computed via the inner product:

$$\text{length}(\vec{v}) = \sqrt{\vec{v} \cdot \vec{v}} .$$

Note that in the two-dimensional case, $\text{length}((v_1, v_2)) = \sqrt{(v_1, v_2) \cdot (v_1, v_2)} = \sqrt{v_1^2 + v_2^2}$, which is exactly the Pythagorean theorem.

Distributivity For any vectors $\vec{v}$, $\vec{w}$, $\vec{y}$, and $\vec{z}$ of the same dimensionality, we have:

$$(\vec{v} + \vec{w}) \cdot (\vec{y} + \vec{z}) = \vec{v} \cdot \vec{y} + \vec{w} \cdot \vec{y} + \vec{v} \cdot \vec{z} + \vec{w} \cdot \vec{z}. \tag{1}$$

As practice, try justifying each step of the following proof of equation 1 for the two-dimensional case:

$$\begin{aligned}
(\vec{v} + \vec{w}) \cdot (\vec{y} + \vec{z}) &= (v_1 + w_1, v_2 + w_2) \cdot (y_1 + z_1, y_2 + z_2) \\
&= (v_1 + w_1)(y_1 + z_1) + (v_2 + w_2)(y_2 + z_2) \\
&= (v_1 y_1 + w_1 y_1 + v_1 z_1 + w_1 z_1) + (v_2 y_2 + w_2 y_2 + v_2 z_2 + w_2 z_2) \\
&= (v_1 y_1 + v_2 y_2) + (w_1 y_1 + w_2 y_2) + (v_1 z_1 + v_2 z_2) + (w_1 z_1 + w_2 z_2) \\
&= \vec{v} \cdot \vec{y} + \vec{w} \cdot \vec{y} + \vec{v} \cdot \vec{z} + \vec{w} \cdot \vec{z}
\end{aligned}$$

Although this all looks analogous to scalar (“single number”) operations, note that there’s no such thing as “vector division”; see the “type-checking” discussion below.

Type-checking In our experience, the following facts can help “sanity check” your work to make sure there aren’t errors.

- The result of adding two vectors (of the same dimensionality) is another vector.
- The result of taking the inner product of two vectors is a scalar (a “single number”).
- The length of a vector is a scalar.
- You can’t take an inner product between two vectors that don’t have the same number of components.
- The $>$ and $<$ inequality signs only apply to scalars.

These rules let us see, for example, that you can’t write something like “$(15, 3) > (1, 5)$”: if you find yourself trying to do so, you might mean $\text{length}((15, 3)) > \text{length}((1, 5))$ or something else.

Another important consequence is that *you cannot “cancel out” vectors by division*. That is, if you have an inequality $\vec{x} \cdot \vec{y} \geq \vec{x} \cdot \vec{z}$, you *cannot* conclude that $\vec{y} \geq \vec{z}$ because, as we just said, an inequality between vectors doesn’t make sense.