Module 17

Recursion

Motivation for Video

- This series is **not** about a control structure
- **Recursion:** a *programming technique*
  - Uses techniques you know in an usual way
  - Duplicates the iteration of for and while
  - Exists because it is often more efficient
- It is a very **advanced** topic
  - You will study this all four years of a CS program
  - We are not expecting you to master this
  - We just want you to understand the foundations

Recursive Definition

- A definition defined in terms of itself
- **Example:** PIP
  - Tool for installing Python packages
  - **PIP** stands for “**PIP** Installs Packages”
- Sounds like a circular definition
  - The example above is
  - But need not be in right circumstances

Example: Factorial

- Non-recursive definition (n an int >= 0):

  $n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$

  $0! = 1$

- Refactor top formula as:

  $n! = n\,(n\text{-}1 \times \ldots \times 2 \times 1)$

- Recursive definition:

  $n! = n\,(n\text{-}1)!$ for $n > 0$ **Recursive case**

  $0! = 1$ **Base case**

Example: Fibonnaci

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  $a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$

  - Refer to element at position n as a_n
  - Get the next element by adding previous two
- Recursive definition:
  - $a_n = a_{n-1} + a_{n-2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Example: Fibonnaci

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Refer to element at position
  - C wo
- R
  - $_{n\text{-}2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

While recursion may be *weird*
it is well-defined and not circular

Recursive Functions

- A function that calls itself
  - Inside of body there is a call to itself
  - Very natural for recursive math defs
- **Recall:** Factorial
  - n! = n (n-1)! **Recursive Case**
  - 0! = 1 **Base Case**

Factorial as a Recursive Function

```
def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥ 0 an int"""
    if n == 0:
        return 1

    return n*factorial(n-1)
```

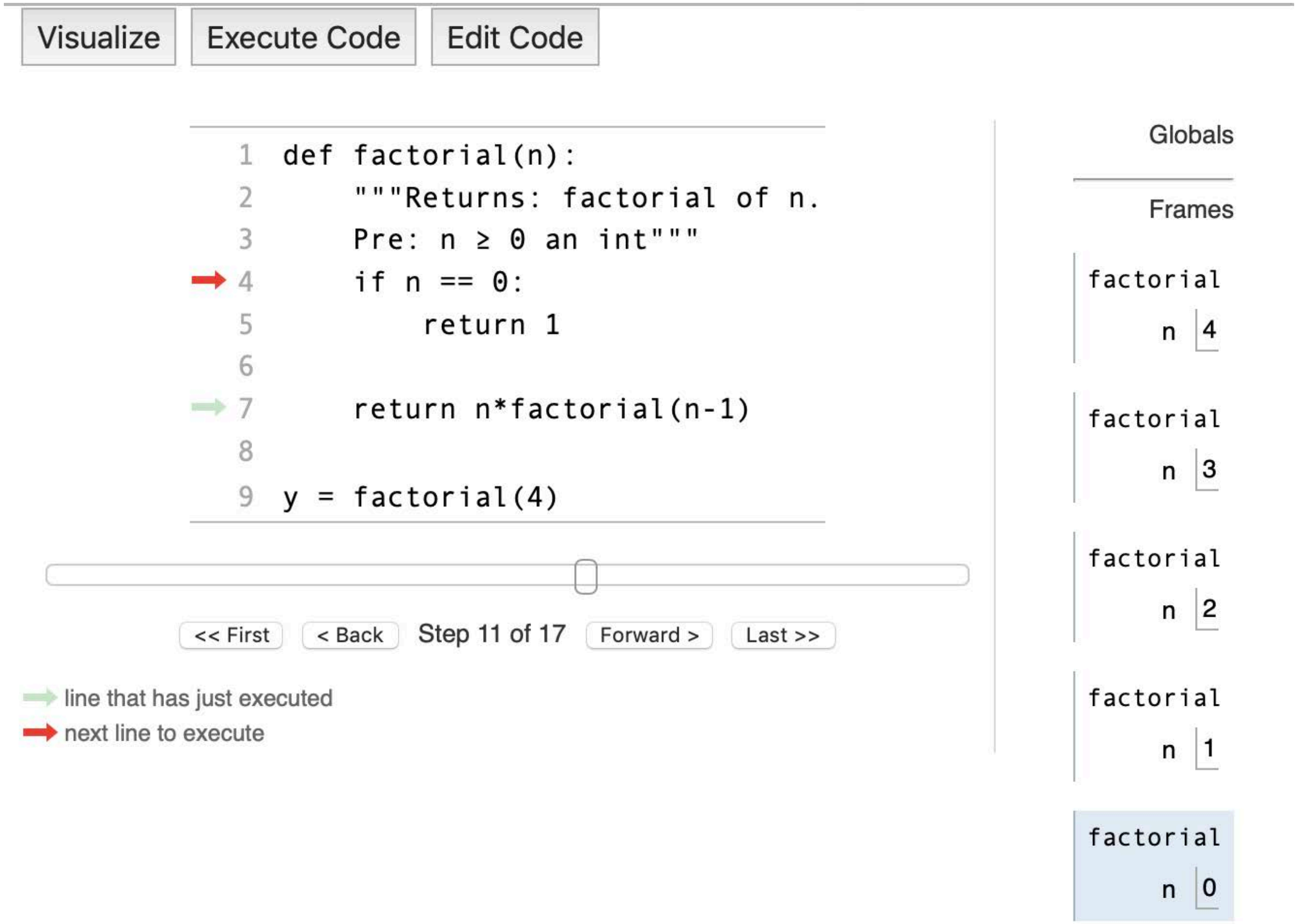
- n! = n (n-1)!
- 0! = 1

Base case(s)

Recursive case

What happens if there is no base case?

Factorial and Call Frames



```
def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥ 0 an int"""
    if n == 0:
        return 1

    return n*factorial(n-1)

y = factorial(4)
```

line that has just executed

next line to execute

factorial
n 4

factorial
n 3

factorial
n 2

factorial
n 1

factorial
n 0

Fibonacci as a Recursive Function

```
def fibonacci(n):
    """Returns: Fibonacci an
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

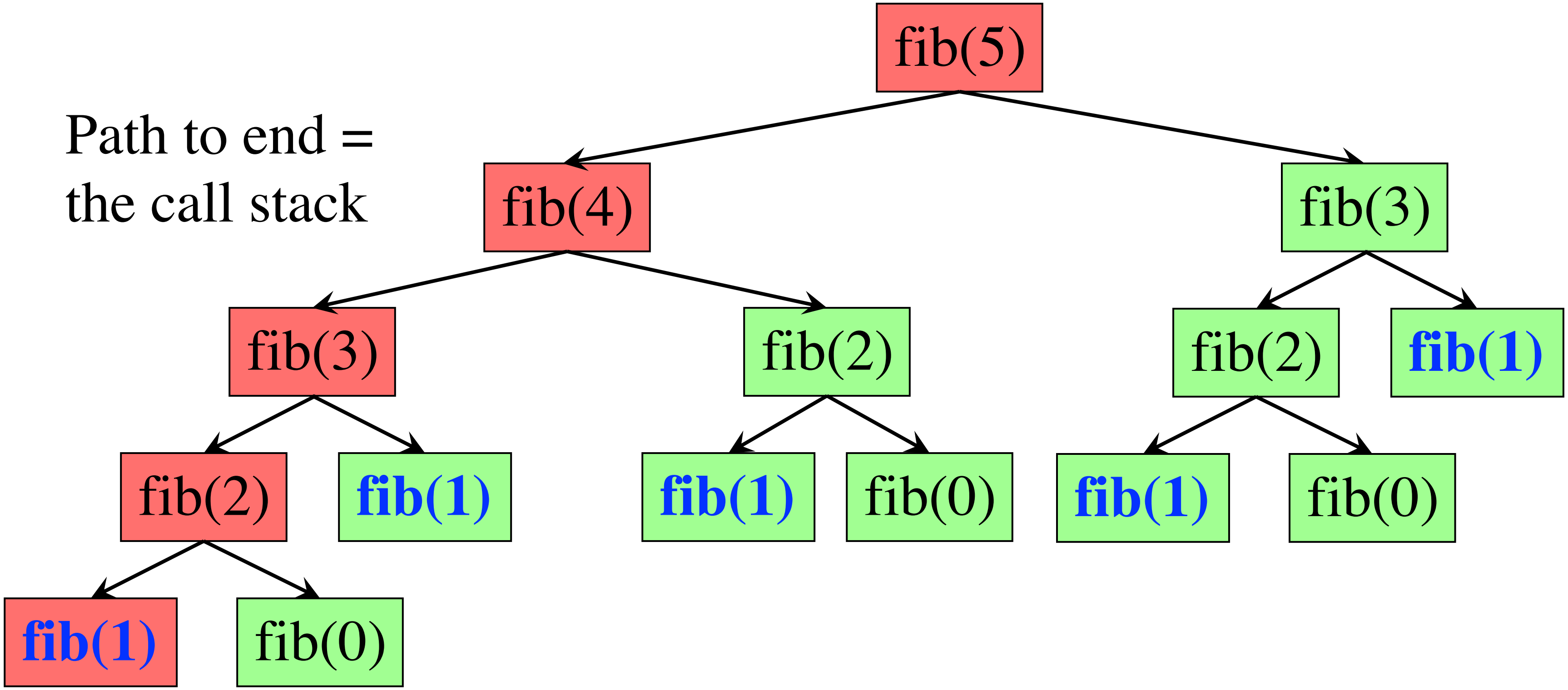
- $a_n = a_{n-1} + a_{n-2}$
- $a_0 = 1$
- $a_1 = 1$

Base case(s)

Recursive case

Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib(n) has a stack that is always ≤ n
  - But fib(n) makes a lot of redundant calls

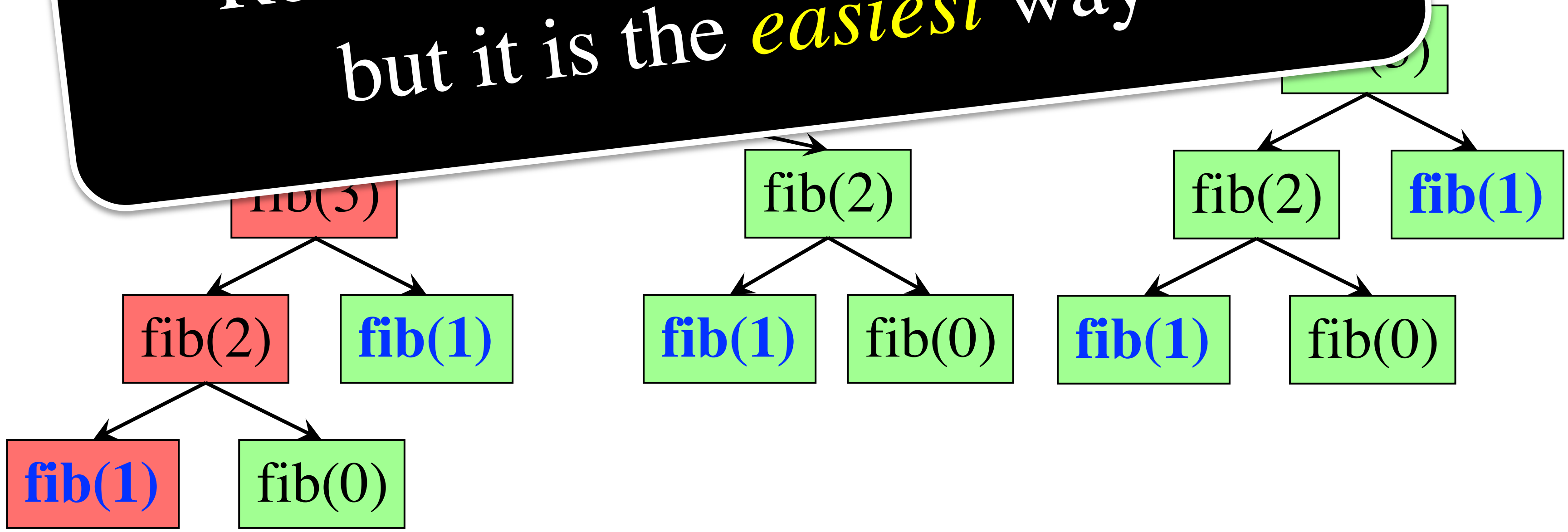


Path to end =
the call stack

Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib(*n*) has a stack that is always ≤ *n*
  - But fib(*n*) makes a lot of redund

Recursion is not the *best* way, but it is the *easiest* way

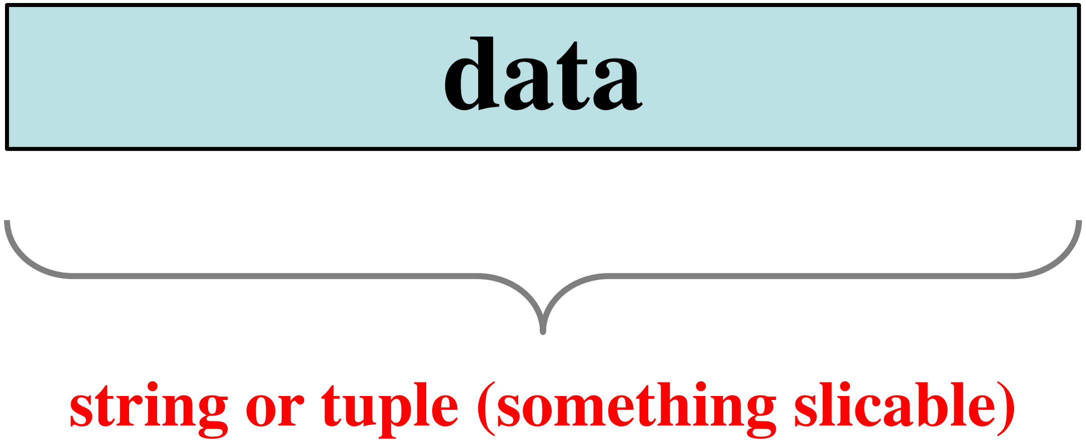


Recursion vs Iteration

- **Recursion** is ***provably equivalent*** to **iteration**
  - Iteration includes **for-loop** and **while-loop** (later)
  - Anything can do in one, can do in the other
- But some things are easier with recursion
  - And some things are easier with iteration
- Will **not** teach you when to choose recursion
  - This is a topic for more advanced courses
- But we will cover one popular use case

Recursion is best for Divide and Conquer

Goal: Solve problem P on a piece of data



data

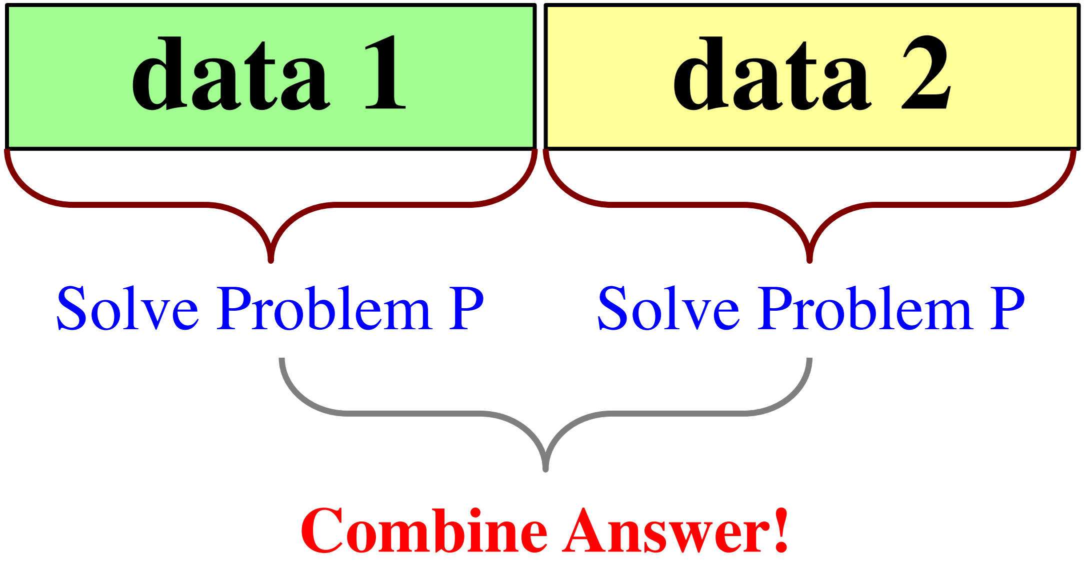
string or tuple (something slicable)

Recursion is best for Divide and Conquer

Goal: Solve problem P on a piece of data

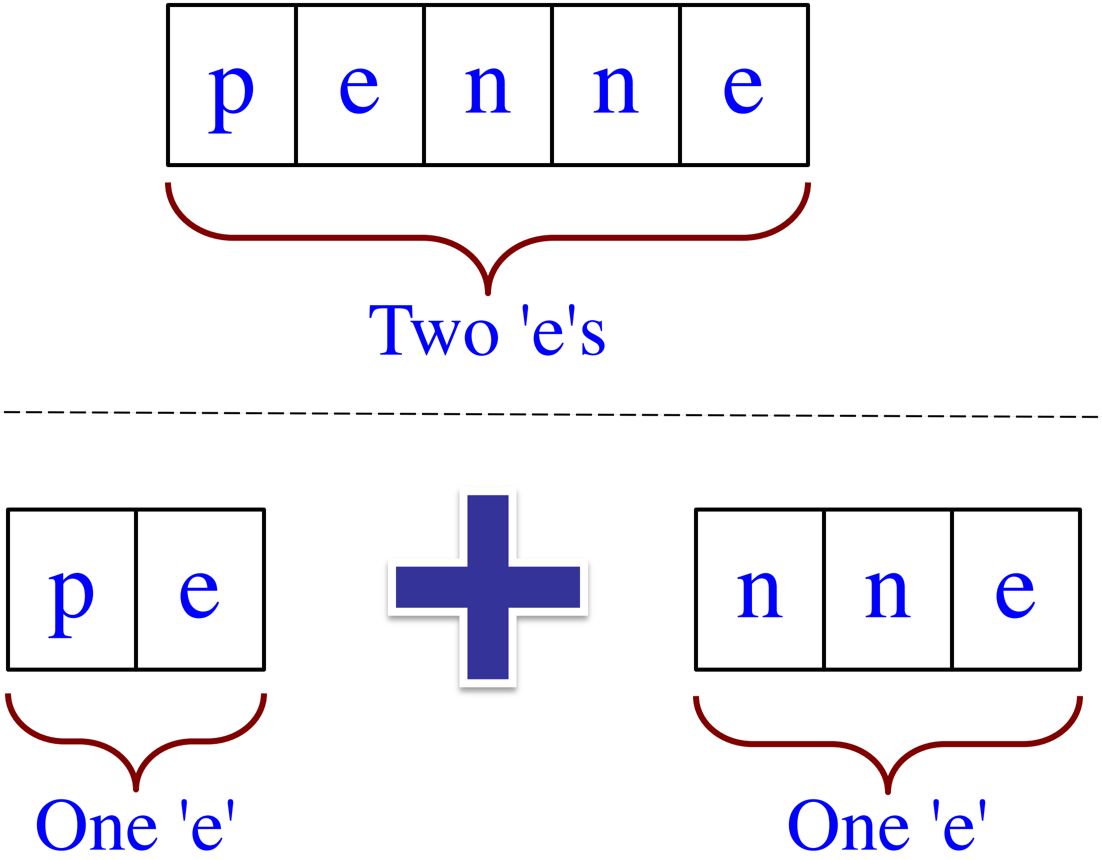
data

Idea: Split data into two parts and solve problem



Divide and Conquer Example

Count the number of 'e's in a string:



p	e	n	n	e

Two 'e's

p	e

\+

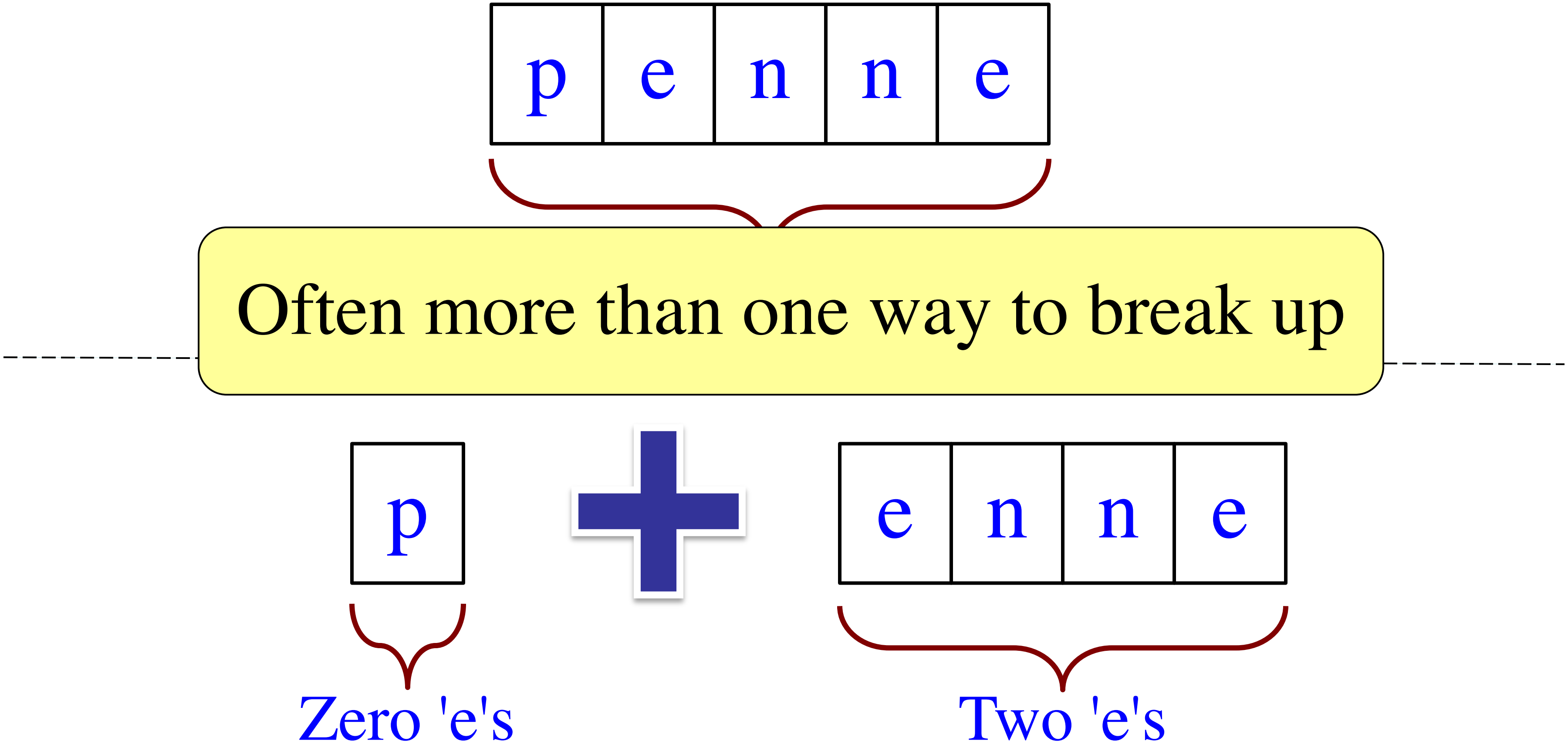
n	n	e

One 'e'

One 'e'

Divide and Conquer Example

Count the number of 'e's in a string:



p e n n e

Often more than one way to break up

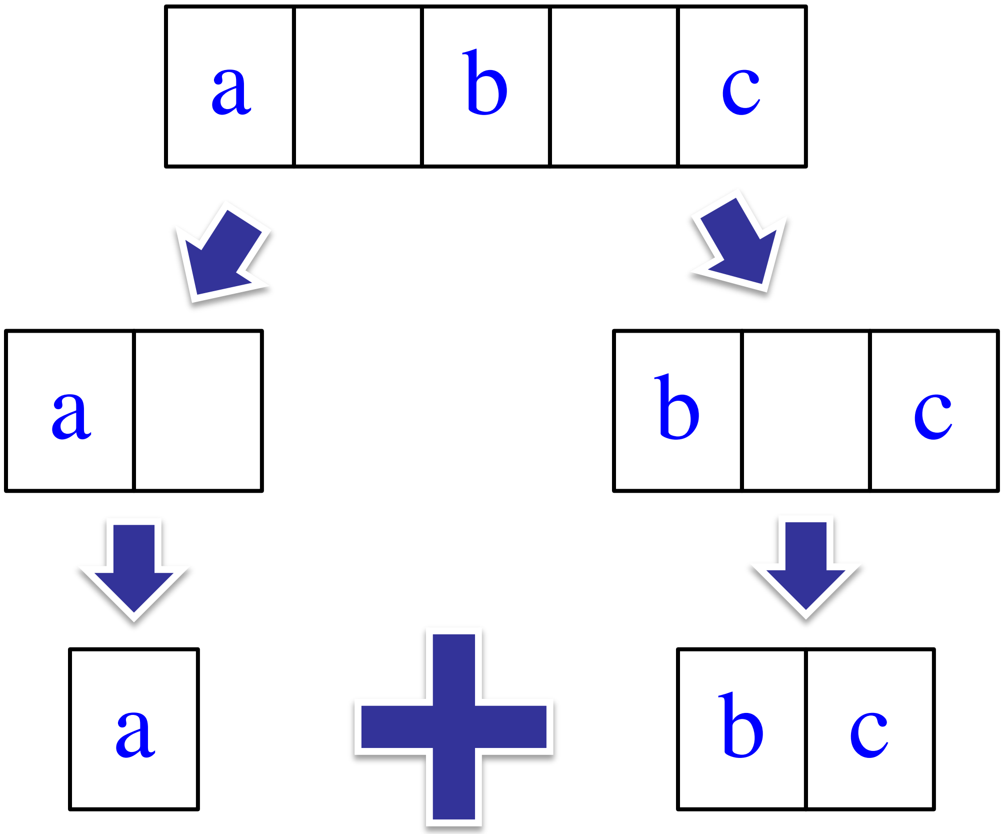
p + e n n e

Zero 'e's

Two 'e's

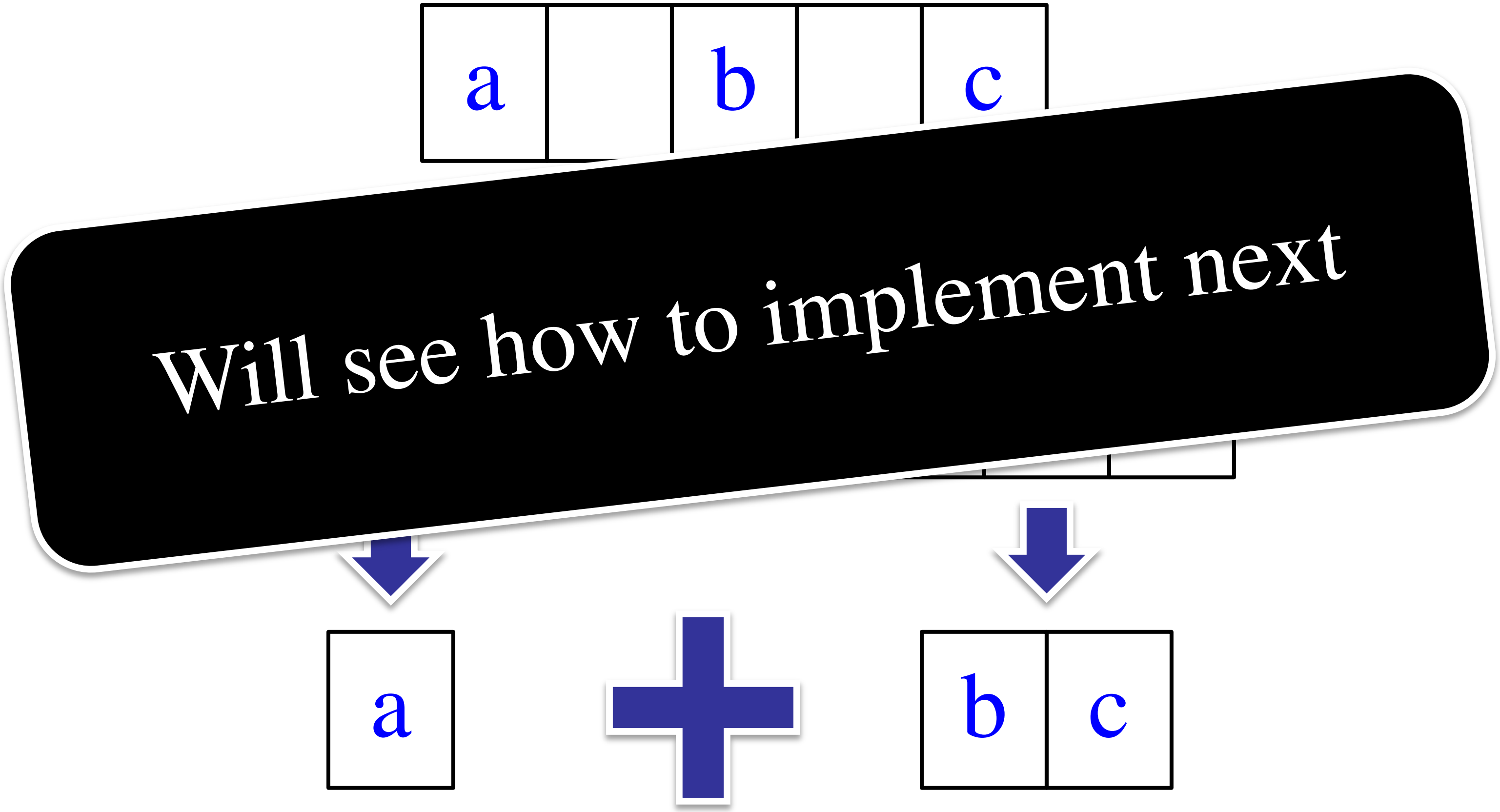
Divide and Conquer Example

Remove all spaces from a string:



Divide and Conquer Example

Remove all spaces from a string:



Will see how to implement next

Three Steps for Divide and Conquer

1. Decide what to do on “small” data
    - Some data cannot be broken up
    - Have to compute this answer directly
2. Decide how to break up your data
    - Both “halves” should be smaller than whole
    - Often no wrong way to do this (next lecture)
3. Decide how to combine your answers
    - Assume the smaller answers are correct
    - Combining them should give bigger answer

Divide and Conquer Example

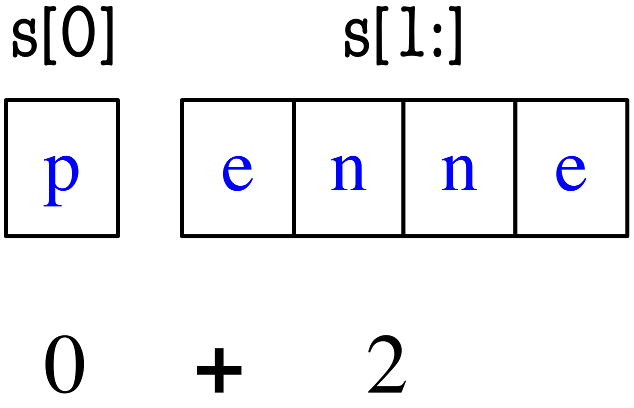
```
def num_es(s):
    """Returns: # of 'e's in s"""
    # 1. Handle small data
    if s == '':
        return 0
    elif len(s) == 1:
        return 1 if s[0] == 'e' else 0

    # 2. Break into two parts
    left = num_es(s[0])
    right = num_es(s[1:])

    # 3. Combine the result
    return left+right
```

“Short-cut” for

```
if s[0] == 'e’:
    return 1
else:
    return 0
```



Divide and Conquer Example

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    # 2. Break into two parts
    left = num_es(s[0])
    right = num_es(s[1:])

    # 3. Combine the result
    return left+right
```

Base Case

Recursive Case

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

1. Decide what to do on “small” data

- If it is the **empty string**, nothing to do

```
if s == '':
    return s
```

- If it is a **single character**, delete it if a blank

```
if s == ' ':     # There is a space here
    return ''  # Empty string
else:
    return s
```

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. Decide how to break it up

```
left = deblank(s[0])      # A string with no blanks
right = deblank(s[1:])    # A string with no blanks
```

3. Decide how to combine the answer

```
return left+right         # String concatenation
```

Putting it All Together

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s
    elif len(s) == 1:
        return '' if s[0] == ' ' else s

    left = deblank(s[0])
    right = deblank(s[1:])

    return left+right
```

Handle small data

Break up the data

Combine answers

Putting it All Together

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def deblank(s):
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    if s == '':
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    left = deblank(s[0])
    right = deblank(s[1:])

    return left+right
```

Handle small data

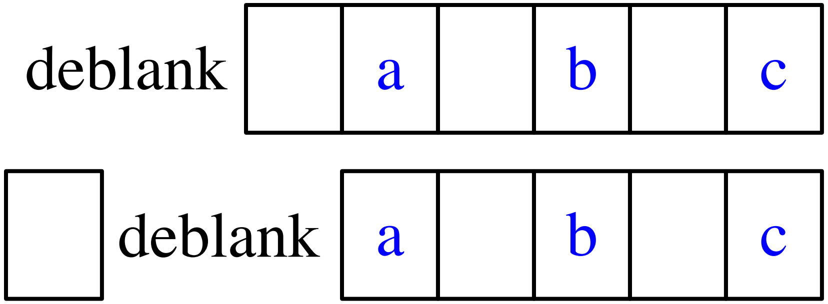
Break up the data

Combine answers

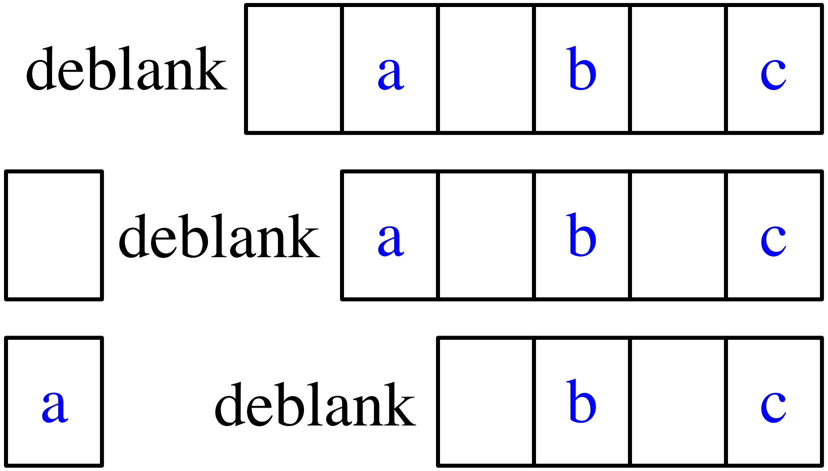
Following the Recursion

deblank		a		b		c

Following the Recursion



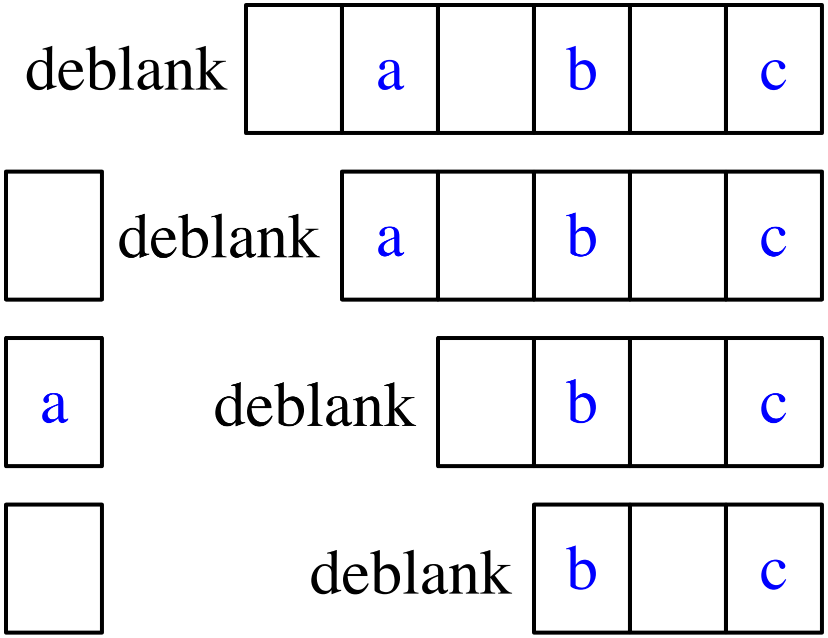
Following the Recursion


deblank a b c

deblank a b c

a deblank b c

Following the Recursion



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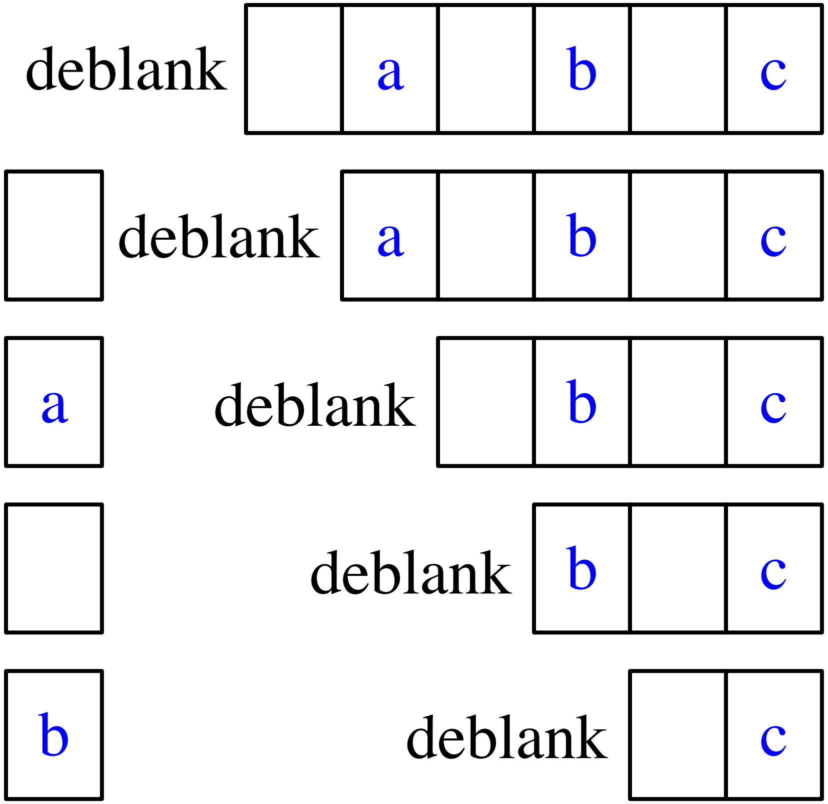


deblank a b c

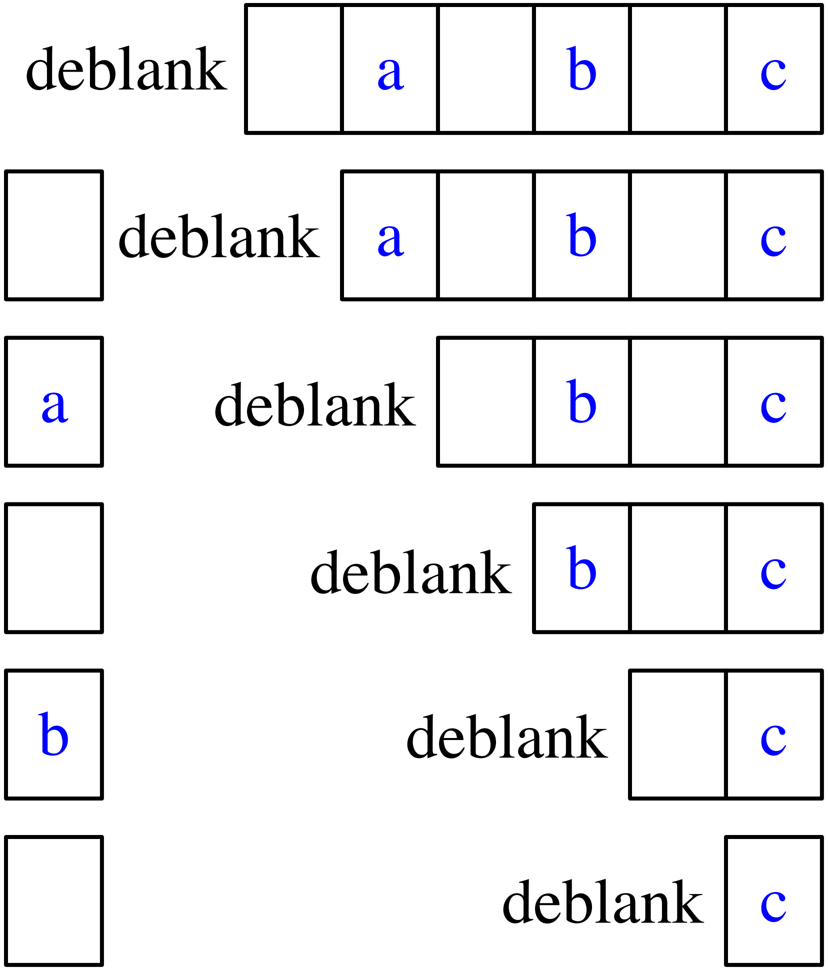
deblank a b c

a deblank b c

deblank b c

b deblank c

Following the Recursion



deblank a b c

deblank a b c

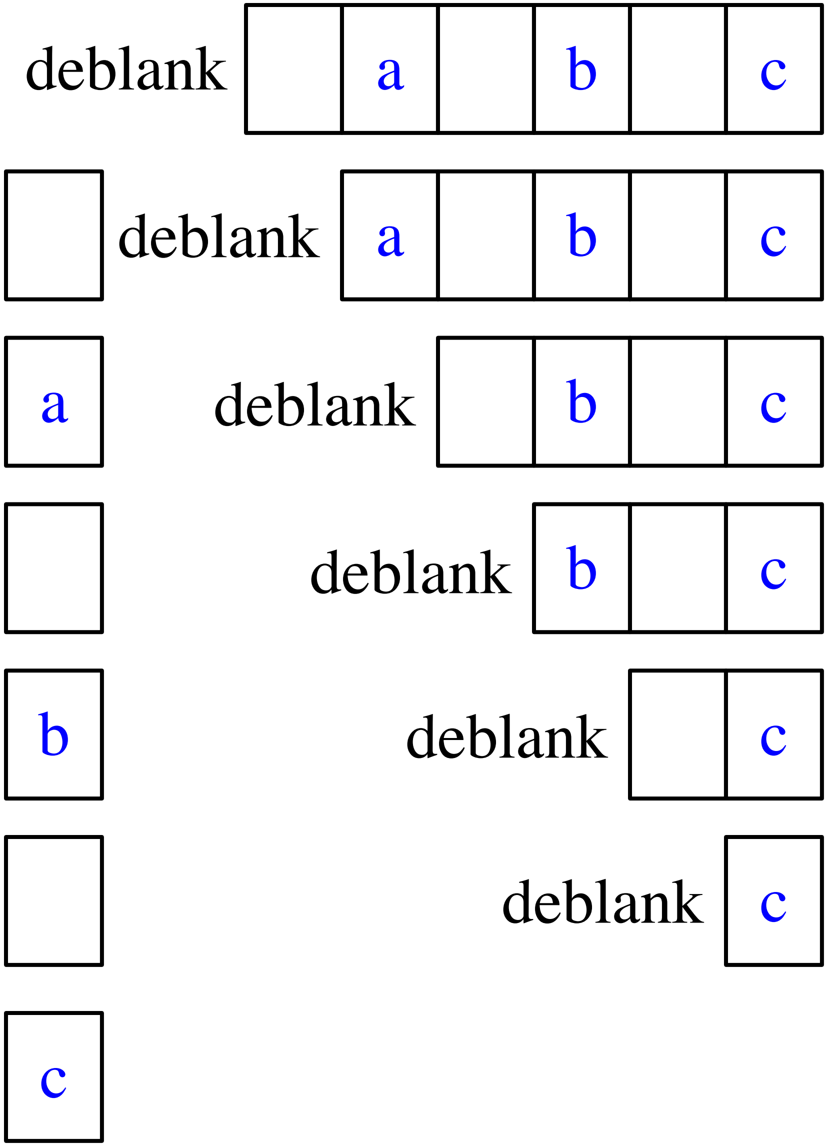
a deblank b c

deblank b c

b deblank c

deblank c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

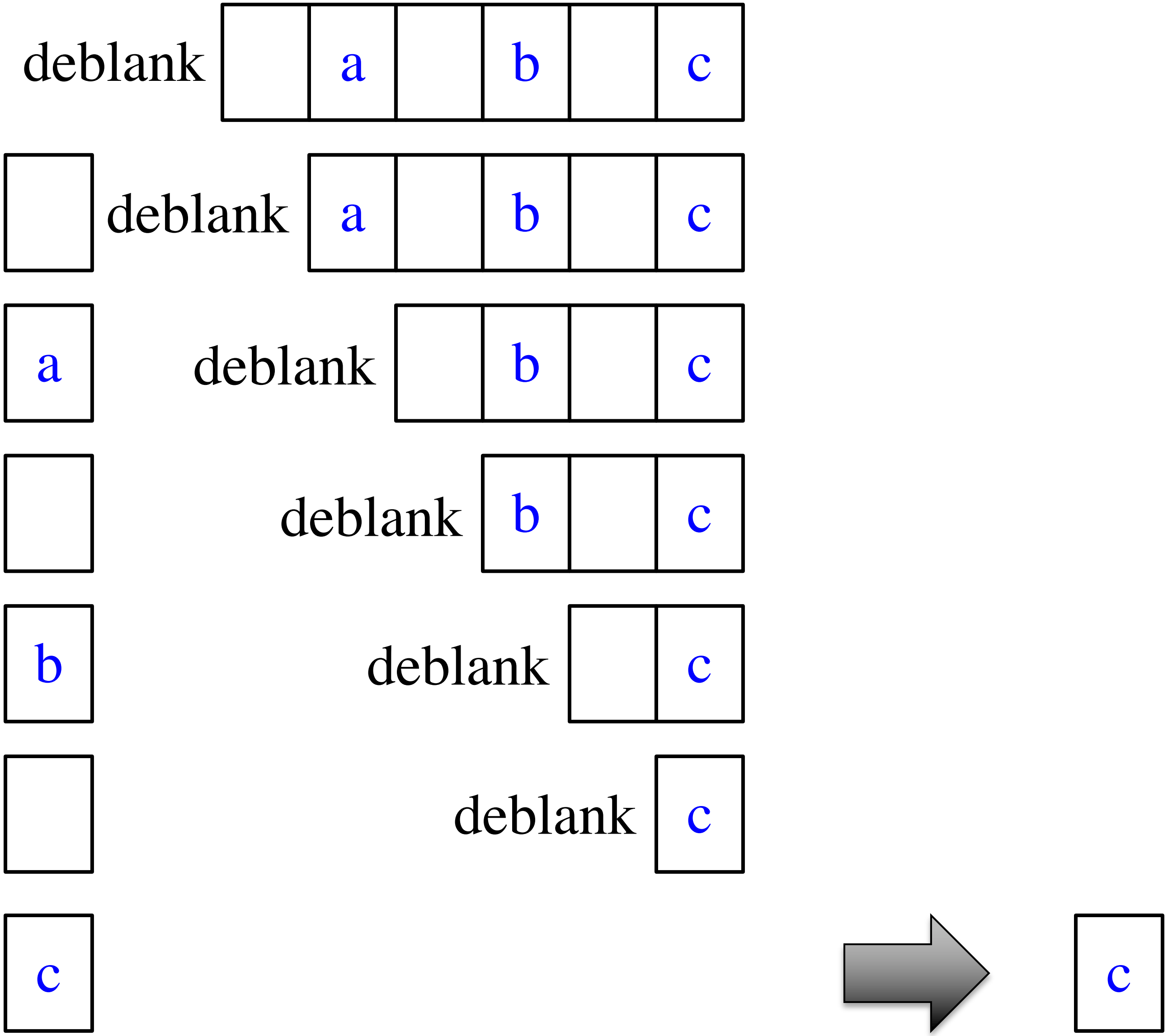
deblank b c

b deblank c

deblank c

c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

deblank b c

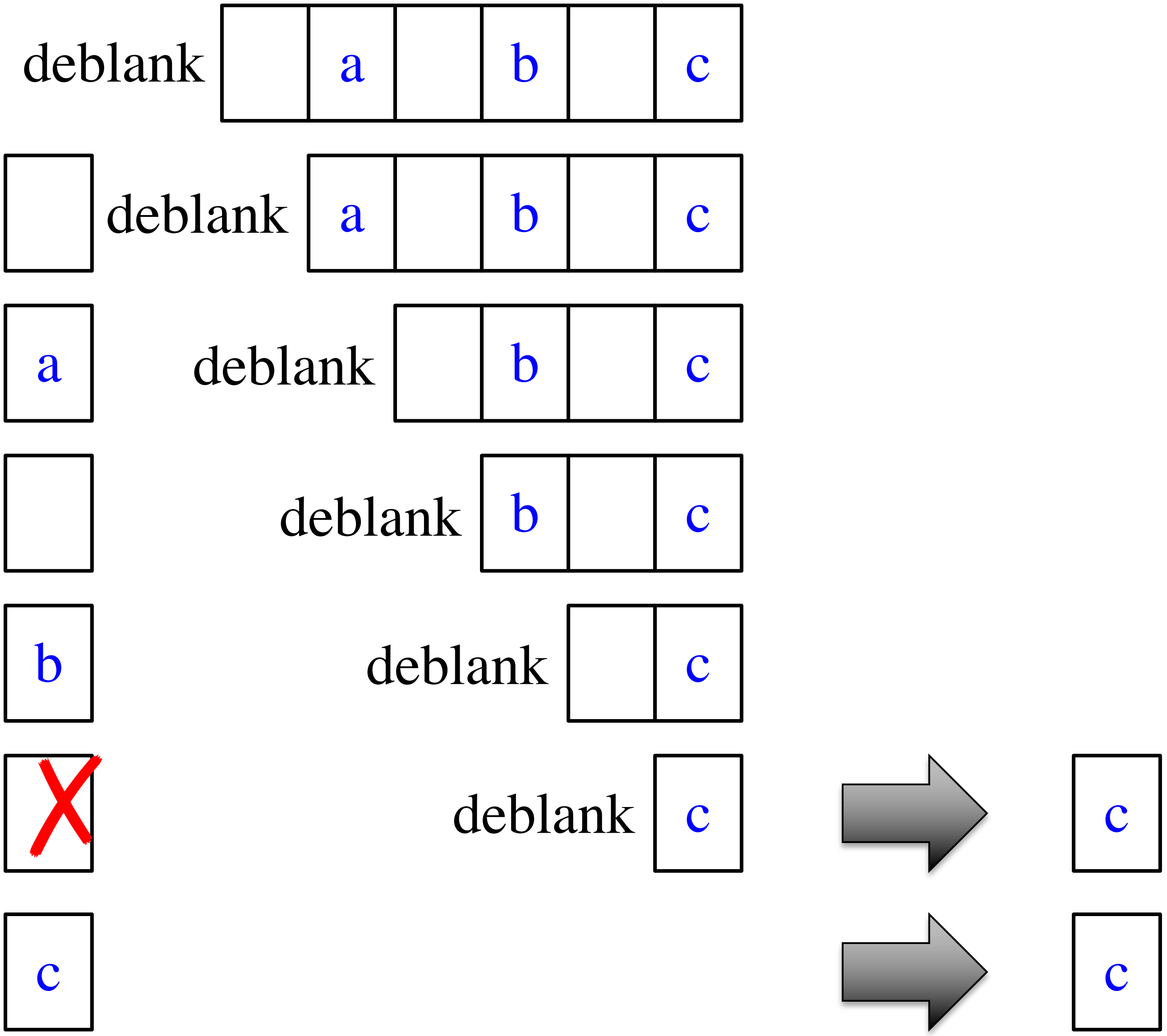
b deblank c

deblank c

c

c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

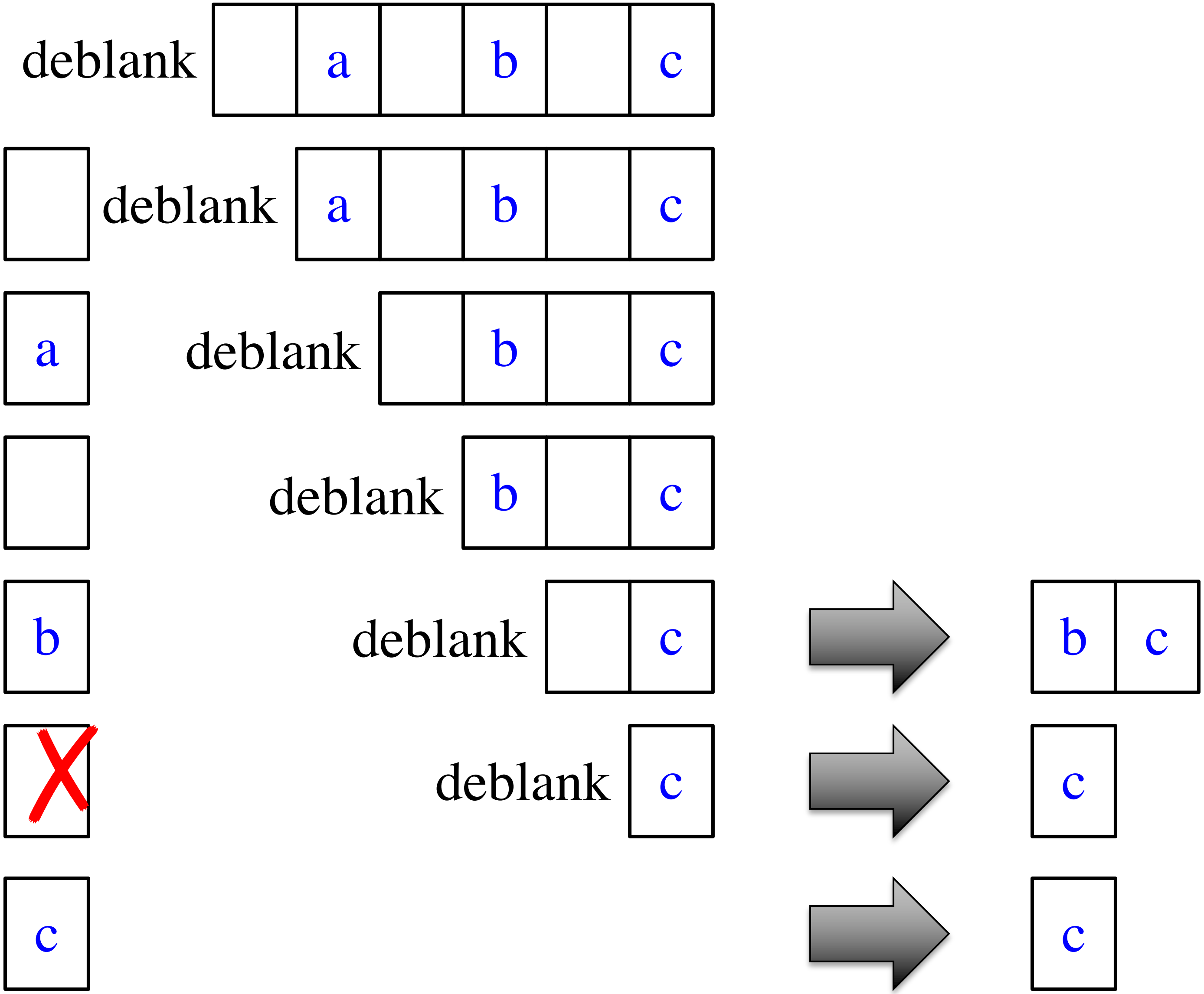
deblank b c

b deblank c

deblank c c

c c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

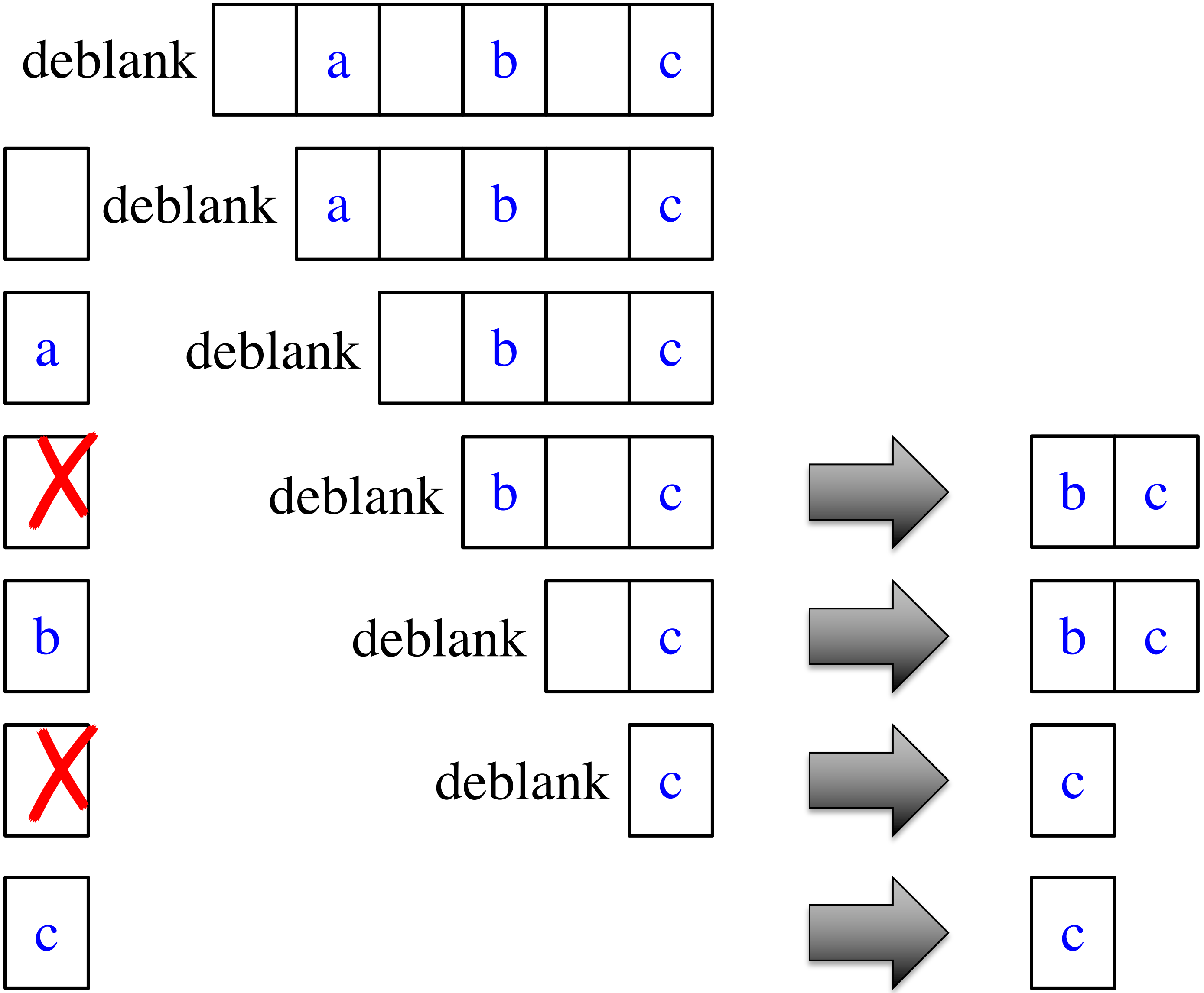
deblank b c

b deblank c b c

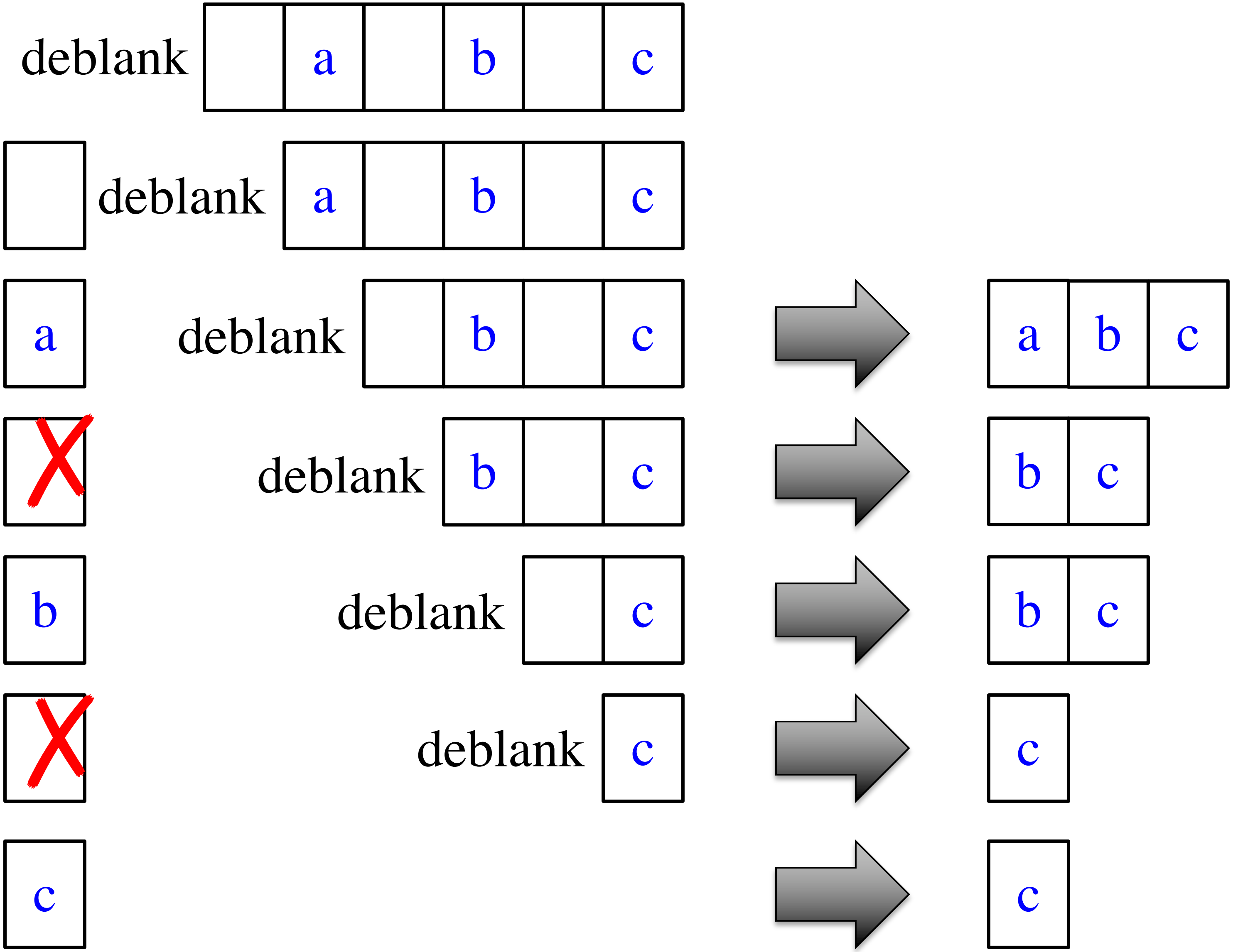
deblank c c

c c

Following the Recursion



Following the Recursion



deblank ␣ a ␣ b ␣ c

␣ deblank a ␣ b ␣ c

a deblank ␣ b ␣ c → a b c

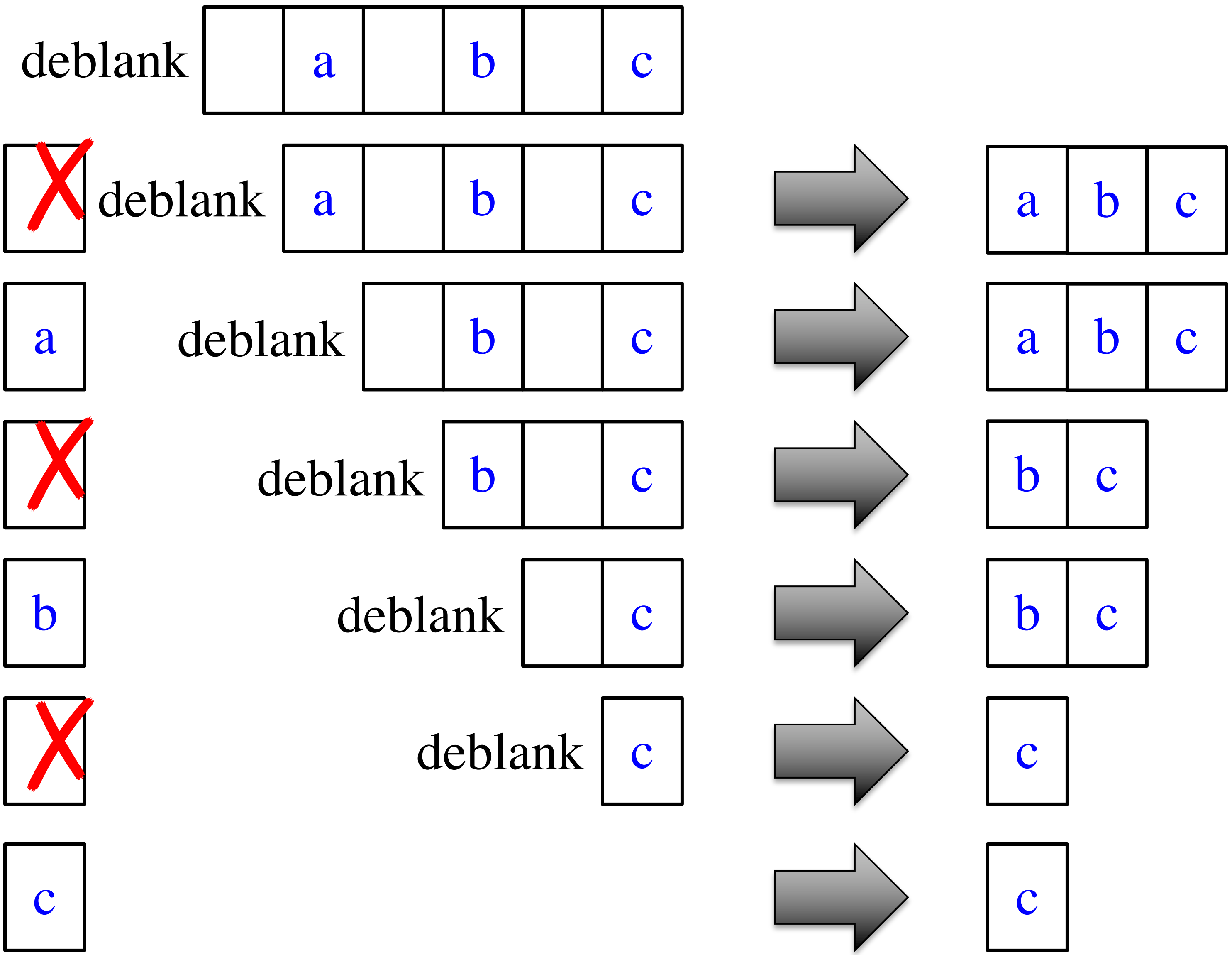
✗ deblank b ␣ c → b c

b deblank ␣ c → b c

✗ deblank c → c

c → c

Following the Recursion



deblank a b c

deblank a b c → a b c

a deblank b c → a b c

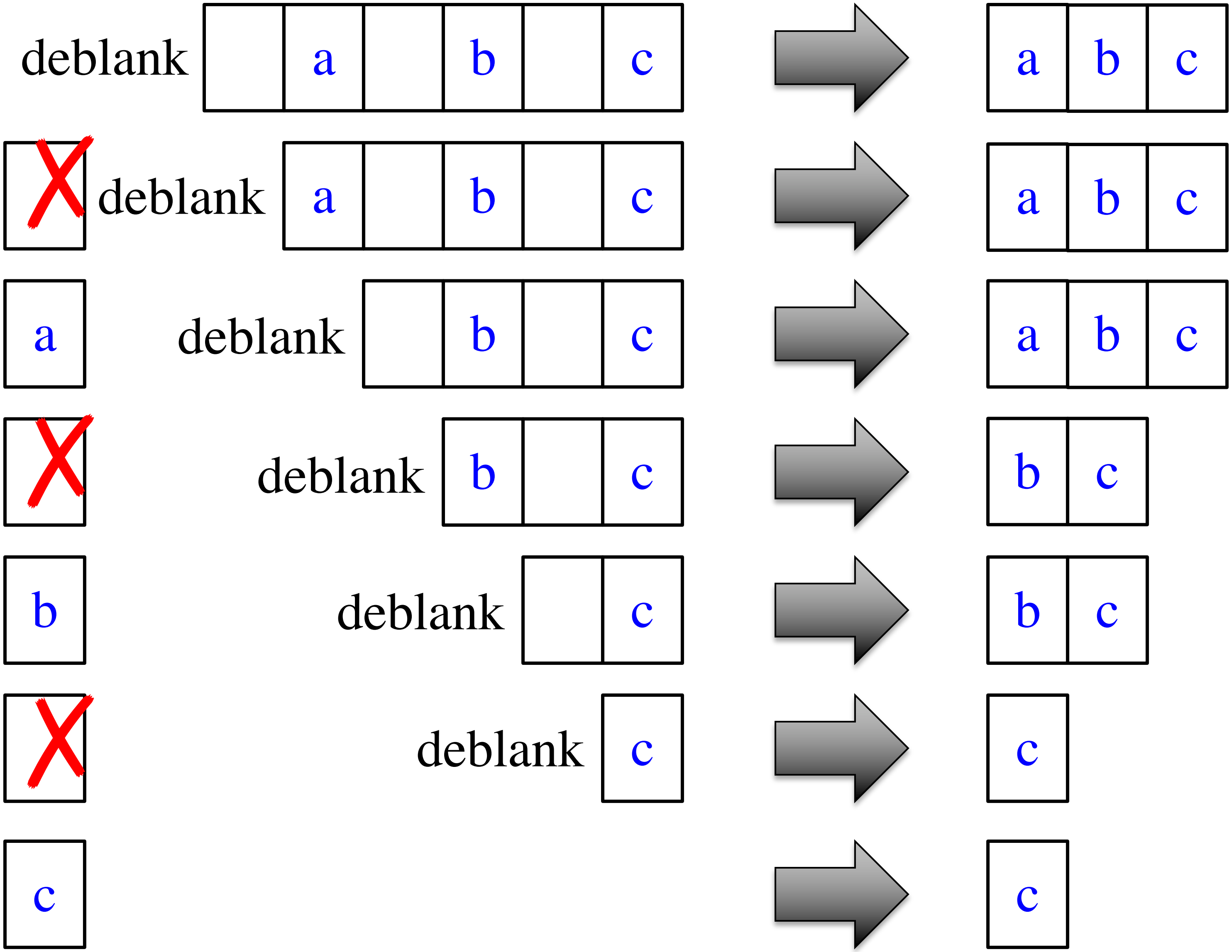
deblank b c → b c

b deblank c → b c

deblank c → c

c → c

Following the Recursion



An Observation

- Divide & Conquer works in phases
  - Starts by splitting the data
  - Gets smaller and smaller
  - Until it reaches the base case
- Only then does it give an answer
  - Gives answer on the small parts
- Then glues all of them back together
  - Glues as the call frames are erased

Recursion vs For-Loop

- Think about our for-loop functions
  - For-loop extract one element at a time
  - Accumulator gathers the return value
- When we have a recursive function
  - The recursive step breaks into single elements
  - The return value IS the accumulator
  - The final step combines the return values
- Divide-and-conquer same as loop+accumulator

Breaking Up Recursion

- D&C requires that we *divide* the data
    - Often does not matter how divide
    - So far, we just pulled off one element
    - **Example:** `'penne'` to `'p'` and `'enne'`
- Can we always do this?
    - It depends on the *combination step*
    - Want to divide to make combination easy

How to Break Up a Recursive Function?

```
def commafy(s):
    """Returns: string with commas every 3 digits
    e.g. commafy('5341267') = '5,341,267'
    Precondition: s represents a non-negative int"""
```

Approach 1

5

341267

How to Break Up a Recursive Function?

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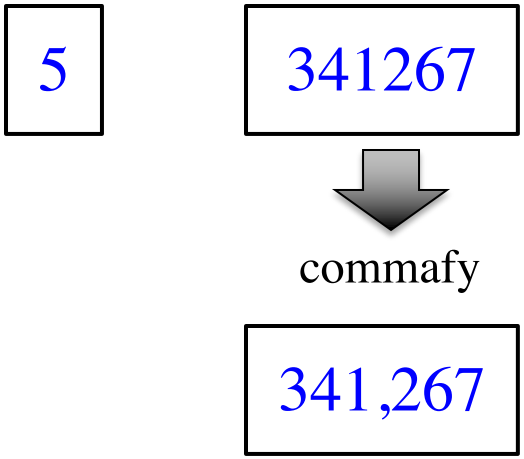

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commafy

341,267

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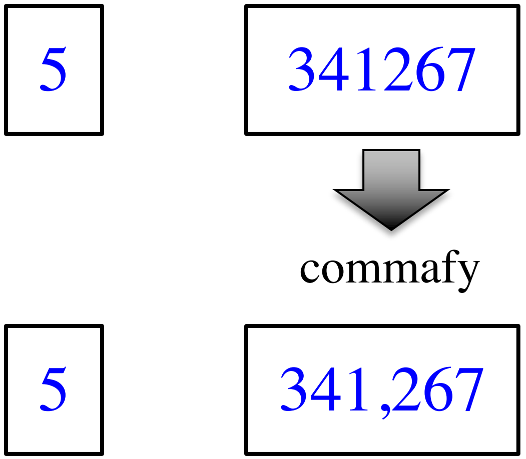

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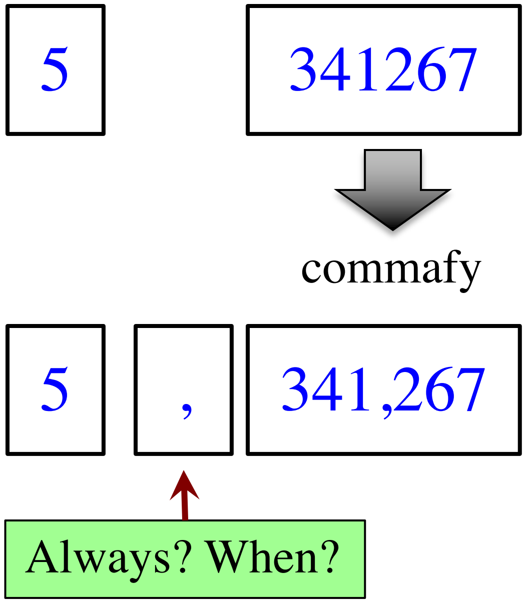

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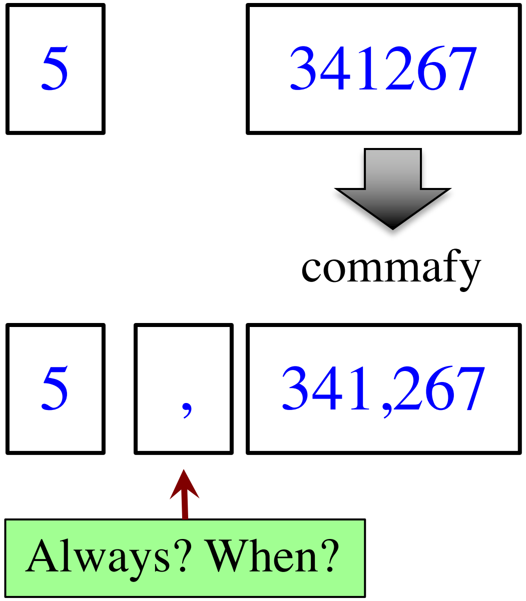

Approach 2

5341

267

How to Break Up a Recursive Function?

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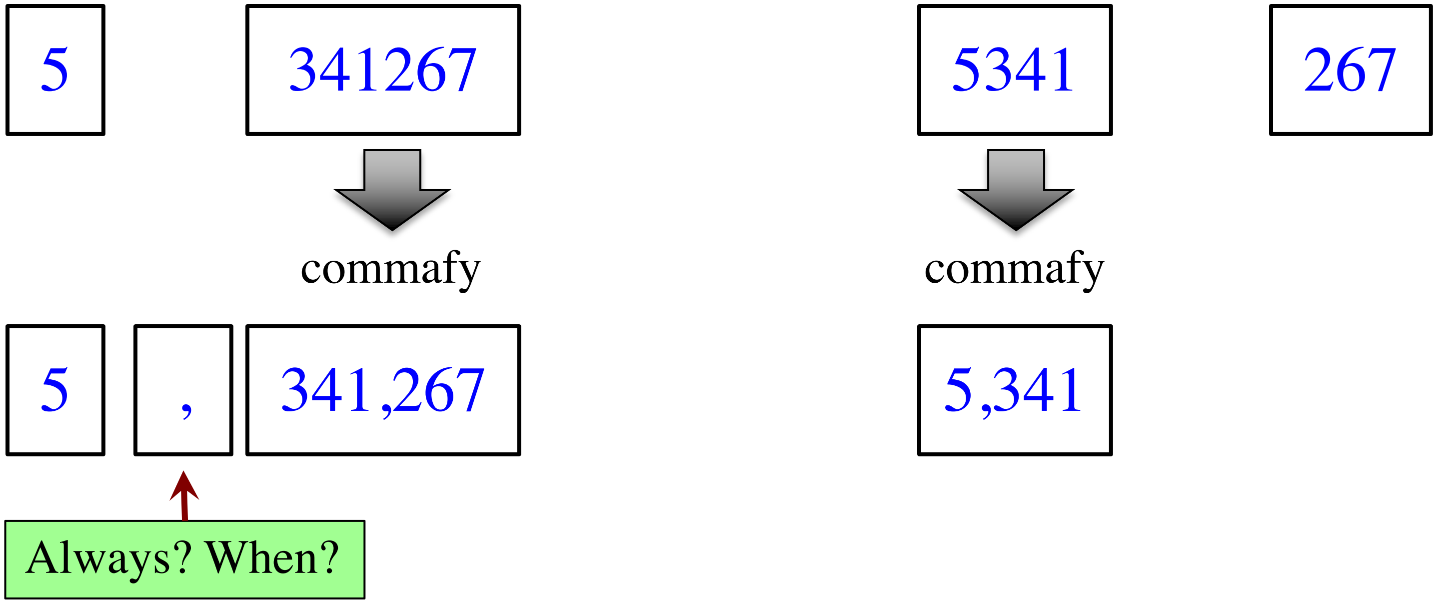
```

Approach 1

Approach 2



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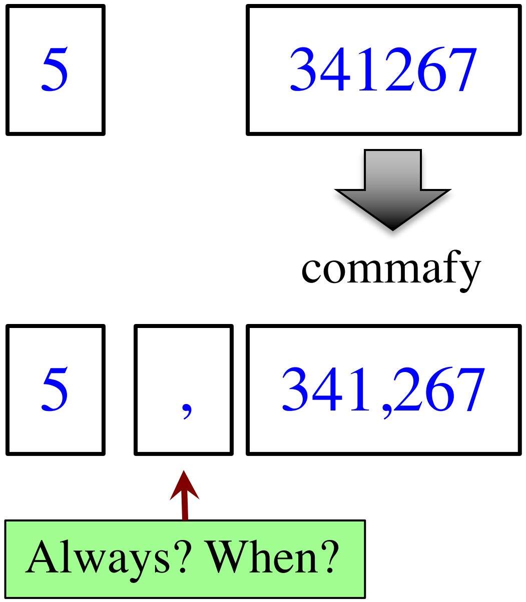

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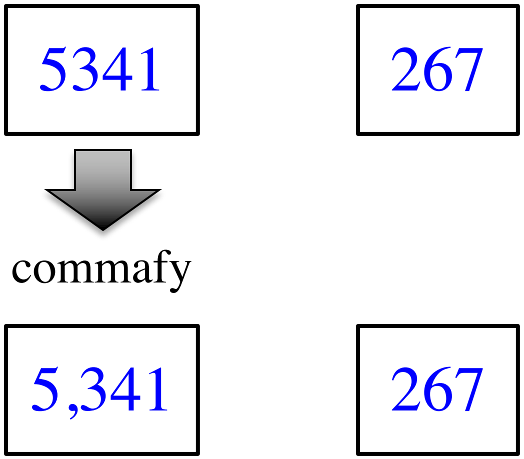
Always? When?

Approach 2



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commafy

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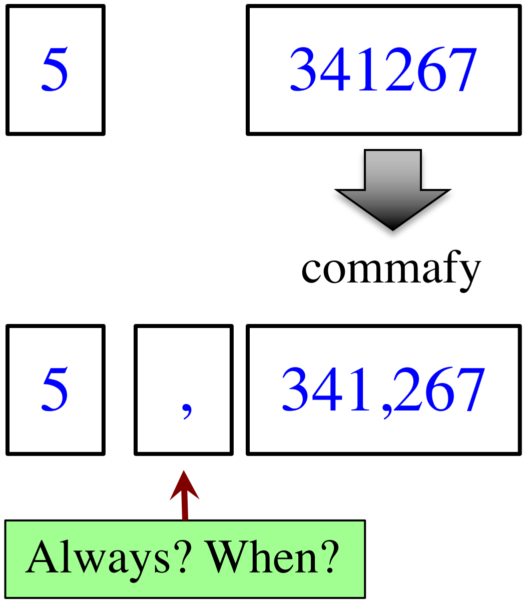

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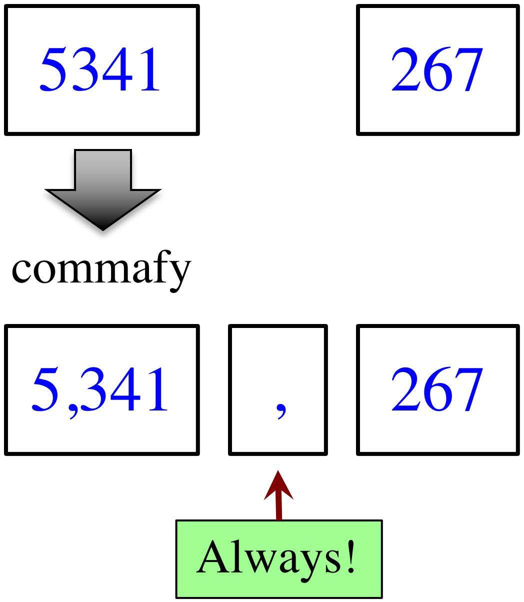
Always? When?

Approach 2



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commafy

5,341

,

267

Always!

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    e.g. commafy('5341267') = '5,341,267'
    Precondition: s represents a non-negative int"""
    # 1. Handle small data.
    if len(s) <= 3:
        return s
```

Base Case

```
    # 2. Break into two parts
    left  = commafy(s[:-3])
    right = s[-3:]  # Small part on RIGHT

    # 3. Combine the result
    return left + ',' + right
```

Recursive Case

More Reasons to be Careful

- Does division only affect code complexity?
    - Does it matter if we are “good” at coding?
    - What if also affects performance?
- Think about the number of recursive calls
    - Each call generates a call frame
    - Have to execute steps in definition (again)
    - So more calls == slower performance
- Want to reduce number of recursive calls

How to Break Up a Recursive Function?

```
def exp(b, c)
    """Returns: b^c
    Precondition: b a float, c ≥ 0 an int"""
```

Approach 1

$$12^{256} = 12 \times (\underbrace{\boxed{12^{255}}}_{\text{Recursive}})$$

$$b^{c} = b \times (b^{c-1})$$

Approach 2

$$12^{256} = (\underbrace{\boxed{12^{128}}}_{\text{Recursive}}) \times (\underbrace{\boxed{12^{128}}}_{\text{Recursive}})$$

$$b^{c} = (b \times b)^{c/2} \text{ if c even}$$

Raising a Number to an Exponent

Approach 1

```
def exp(b, c)
    """Returns: b^c
    Precond: b a float, c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # b^c = b(b^(c-1))
    left = b
    right = exp(b,c-1)

    return left*right
```

Approach 2

```
def exp(b, c)
    """Returns: b^c
    Precond: b a float, c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # c > 0
    if c % 2 == 0:
        return exp(b*b,c//2)

    return b*exp(b*b,(c-1)//2)
```

Raising a Number to an Exponent

Approach 1

```
def exp(b, c)
    """Returns: $b^c$
    Precond: b a float, c ≥ 0 an int"""
    # $b^0$ is 1
    if c == 0:
        return 1

    # $b^c = b(b^{c-1})$
    left = b
    right = exp(b,c-1)

    return left*right
```

Approach 2

```
def exp(b, c)
    """Returns: $b^c$
    Precond: b a float, c ≥ 0 an int"""
    # $b^0$ is 1
    if c == 0:
        return 1

    # c > 0
    if c % 2 == 0
        return exp(b*b,c//2)

    return b*exp(b*b,(c-1)//2)
```

left

right

left

right

Raising a Number to an Exponent

```
def exp(b, c)
    """Returns: b^c
    Precond: b a float, c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # c > 0
    if c % 2 == 0:
        return exp(b*b,c//2)
    return b*exp(b*b,(c-1)//2)
```

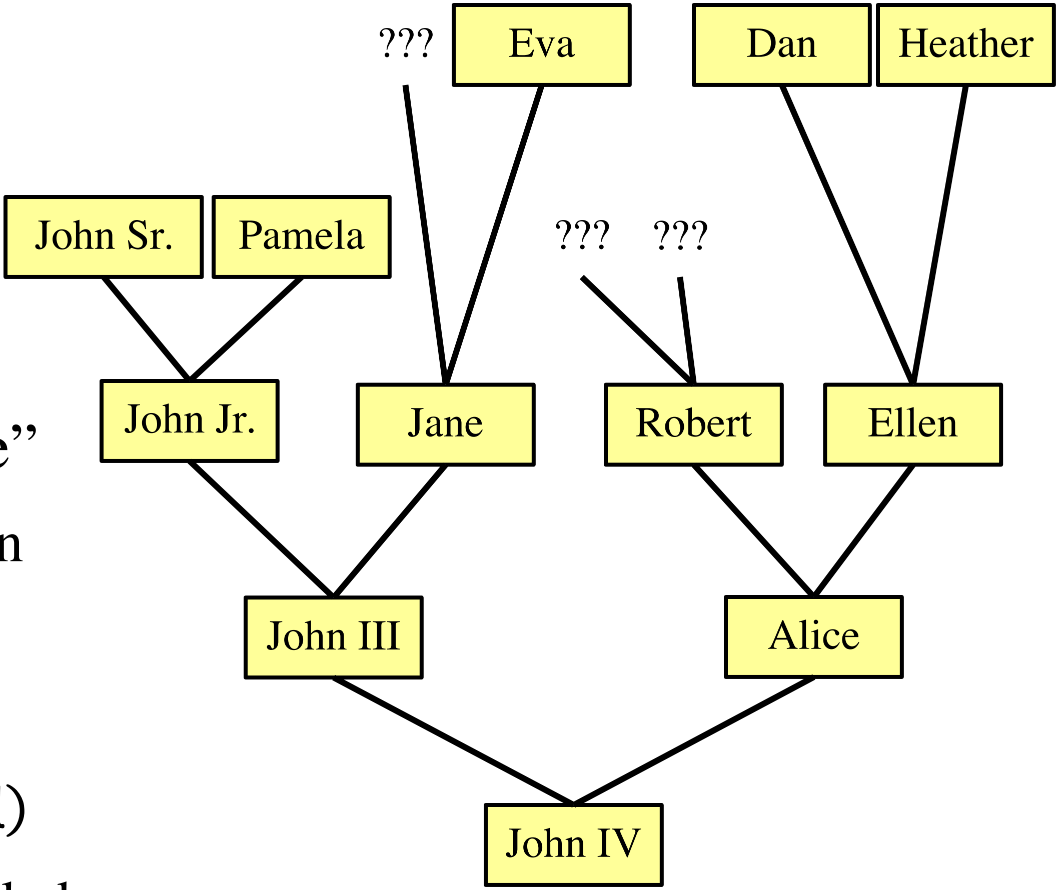
c	# of calls
0	0
1	1
2	2
4	3
8	4
16	5
32	6
2^n	n + 1

32768 is 215

b^{32768} needs only 215 calls!

Recursion and Objects

- Class Person (`person.py`)
  - Objects have 3 attributes
  - `name`: String
  - `mom`: Person (or `None`)
  - `dad`: Person (or `None`)
- Represents the "family tree"
  - Goes as far back as known
  - Attributes `mom` and `dad` are `None` if not known
- **Constructor**: `Person(n,m,d)`
  - Or `Person(n)` if no mom, dad



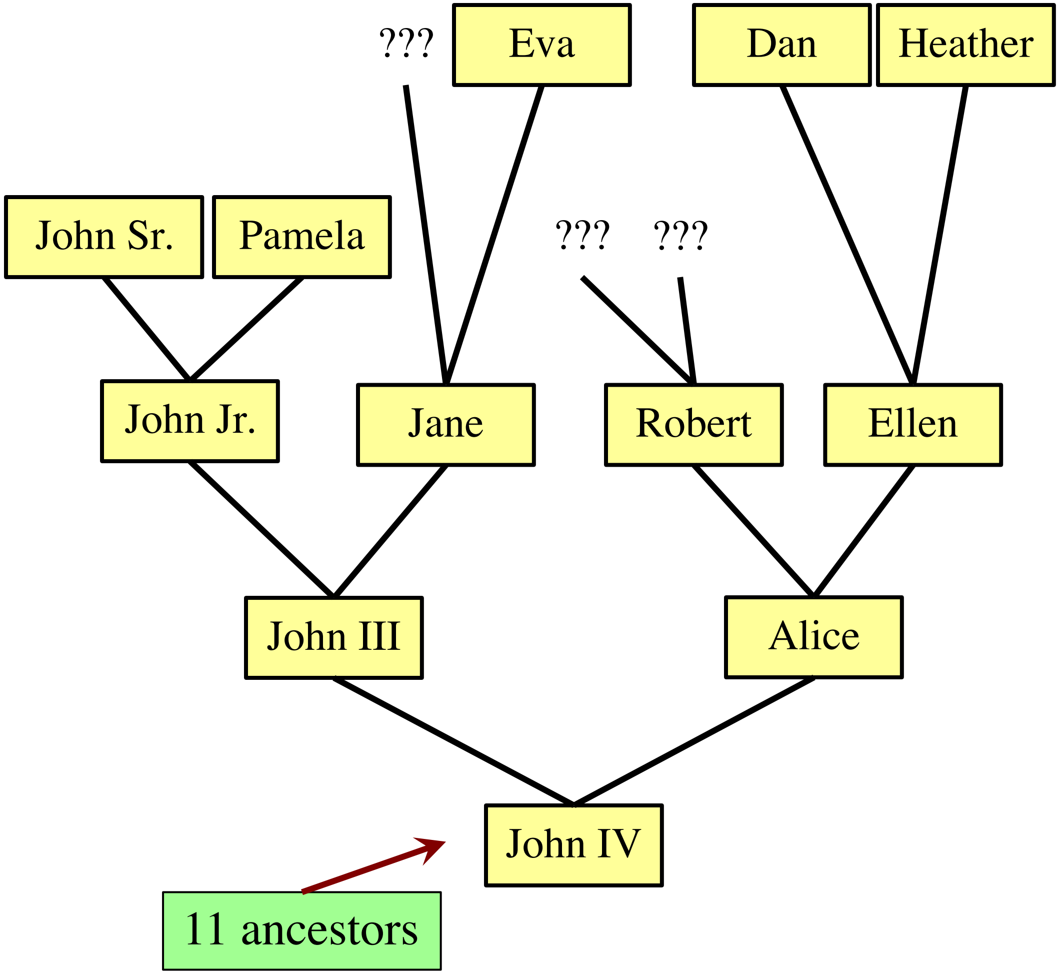
Recursion and Objects

```
def num_ancestors(p):
    """Returns: num of known ancestors
    Pre: p is a Person"""
    # 1. Handle small data.
    # No mom or dad (no ancestors)


    # 2. Break into two parts
    # Has mom or dad
    # Count ancestors of each one
    # (plus mom, dad themselves)



    # 3. Combine the result
```

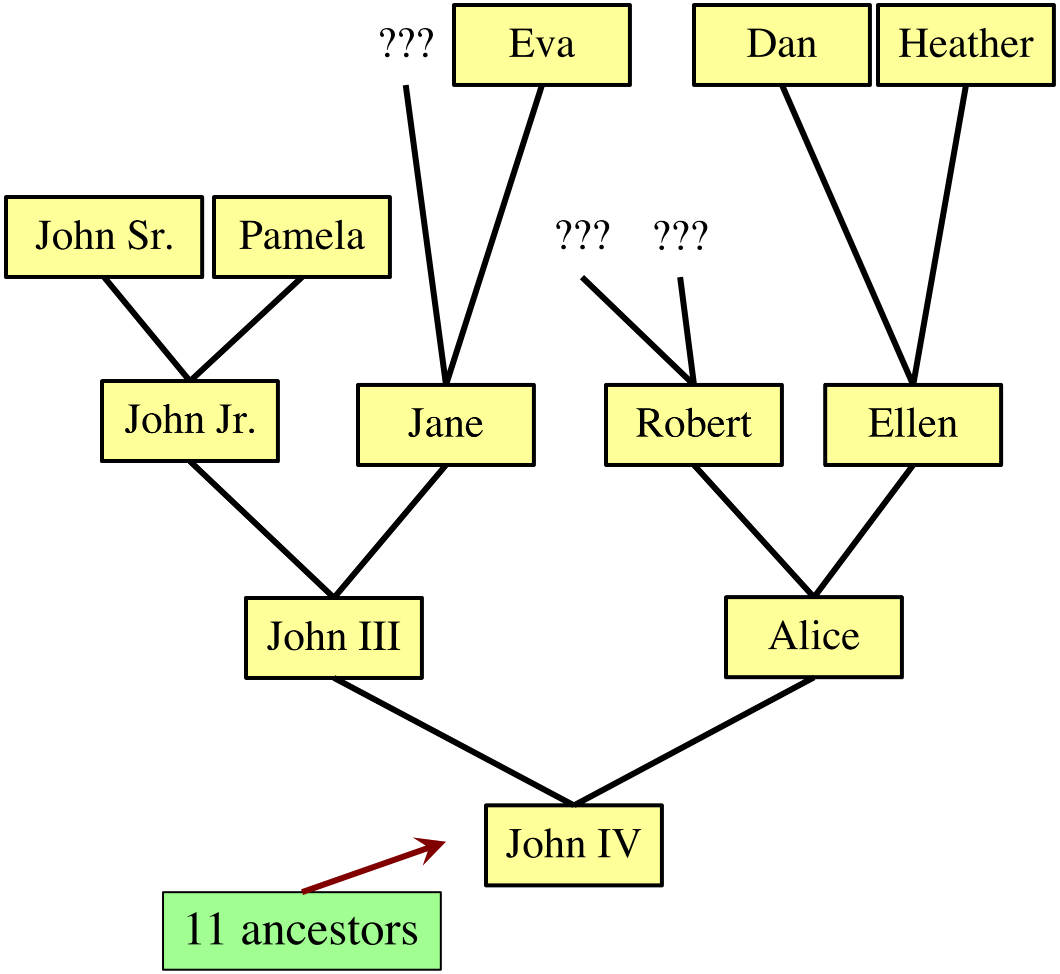


Recursion and Objects

```
def num_ancestors(p):
    """Returns: num of known ancestors
    Pre: p is a Person"""
    # 1. Handle small data.
    if p.mom == None and p.dad == None:
        return 0

    # 2. Break into two parts
    moms = 0
    if not p.mom == None:
        moms = 1+num_ancestors(p.mom)
    dads = 0
    if not p.dad== None:
        dads = 1+num_ancestors(p.dad)

    # 3. Combine the result
    return moms+dads
```



Is All Recursion Divide and Conquer?

- Divide and conquer implies two halves "equal"
  - Performing the same check on each half
  - With some optimization for small halves
- Sometimes we are given a **recursive definition**
  - Math formula to compute that is recursive
  - String definition to check that is recursive
  - Picture to draw that is recursive
  - **Example**: $n! = n\,(n-1)!$
- In that case, we are just implementing definition

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form a palindrome
- **Example:**

have to be the same

AMANAPLANACANALPANAMA

has to be a palindrome

- **Function to Implement:**

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
```

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form a palindrome

Recursive Definition

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
    if len(s) < 2:
        return True

    # Halves not the same; not divide and conquer
    ends = s[0] == s[-1]
    middle = ispalindrome(s[1:-1])
    return ends and middle
```

Base case

Recursive case

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form

But what if we want to *deviate*?

```
def ispalindrome(s):
    """Returns: True if s is a palindrom
    if len(s) < 2:
        return True

    # Halves not the same; not divide and conquer
    ends = s[0] == s[-1]
    middle = ispalindrome(s[1:-1])
    return ends and middle
```

Base case

Recursive case

Recursive Functions and Helpers

```
def ispalindrome2(s):
    """Returns: True if s is a palindrome
    Case of characters is ignored."""
    if len(s) < 2:
        return True
    # Halves not the same; not divide and conquer
    ends = equals_ignore_case(s[0], s[-1])
    middle = ispalindrome(s[1:-1])
    return ends and middle
```

Recursive Functions and Helpers

```
def ispalindrome2(s):
    """Returns: True if s is a palindrome
    Case of characters is ignored."""
    if len(s) < 2:
        return True
    # Halves not the same; not divide and conquer
    ends = equals_ignore_case(s[0], s[-1])
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    return ends and middle
```

Recursive Functions and Helpers

```
def ispalindrome2(s):
    """Returns: True if s is a palindrome
    Case of characters is ignored."""
    if len(s) < 2:
        return True
    # Halves not the same; not divide and conquer
    ends = equals_ignore_case(s[0], s[-1])
    middle = ispalindrome(s[1:-1])
    return ends and middle

def equals_ignore_case(a, b):
    """Returns: True if a and b are same ignoring case"""
    return a.upper() == b.upper()
```

Use helper functions!

- Pull out anything not part of the recursion
- Keeps your code simple and easy to follow

Example: More Palindromes

```
def ispalindrome3(s):
    """Returns: True if s is a palindrome
    Case of characters and non-letters ignored."""
    return ispalindrome2(depunct(s))
```

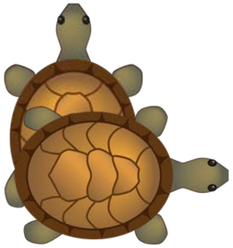
```
def depunct(s):
    """Returns: s with non-letters removed"""
    if s == '':
        return s
    # Combine left and right
    if s[0] in string.letters:
        return s[0]+depunct(s[1:])
    # Ignore left if it is not a letter
    return depunct(s[1:])
```

Use helper functions!

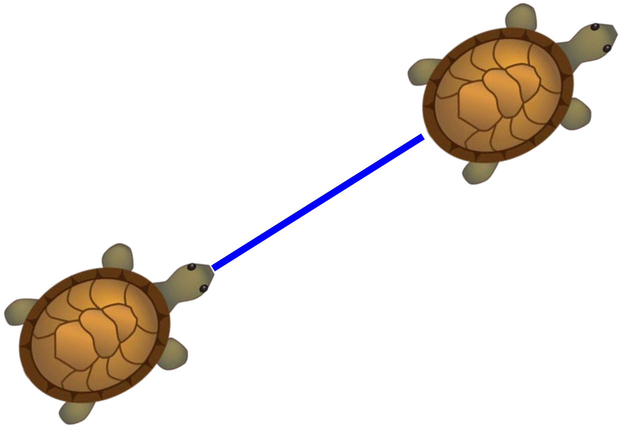
- Sometimes the helper is a recursive function
- Allows you break up problem in smaller parts

“Turtle” Graphics: Assignment A4

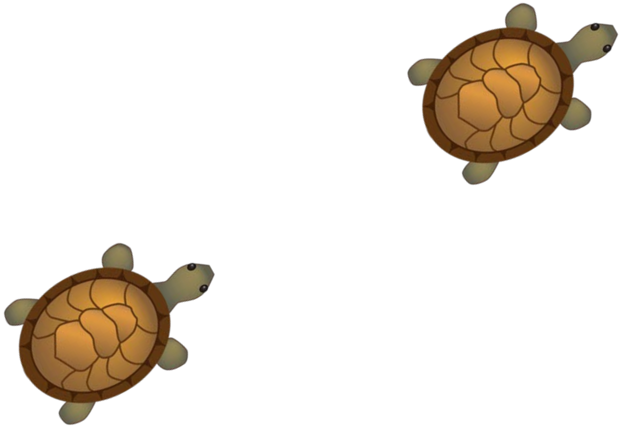
Turn



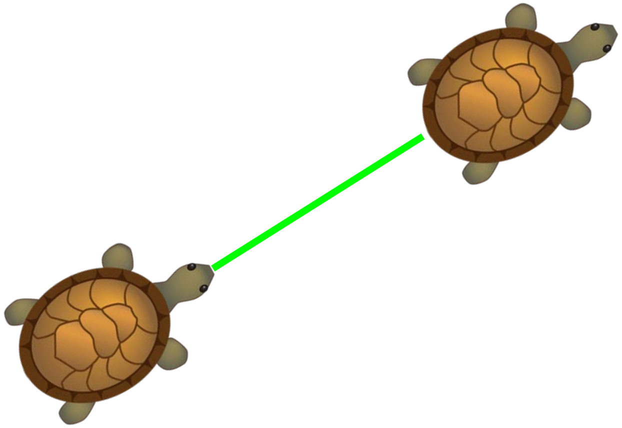
Draw Line



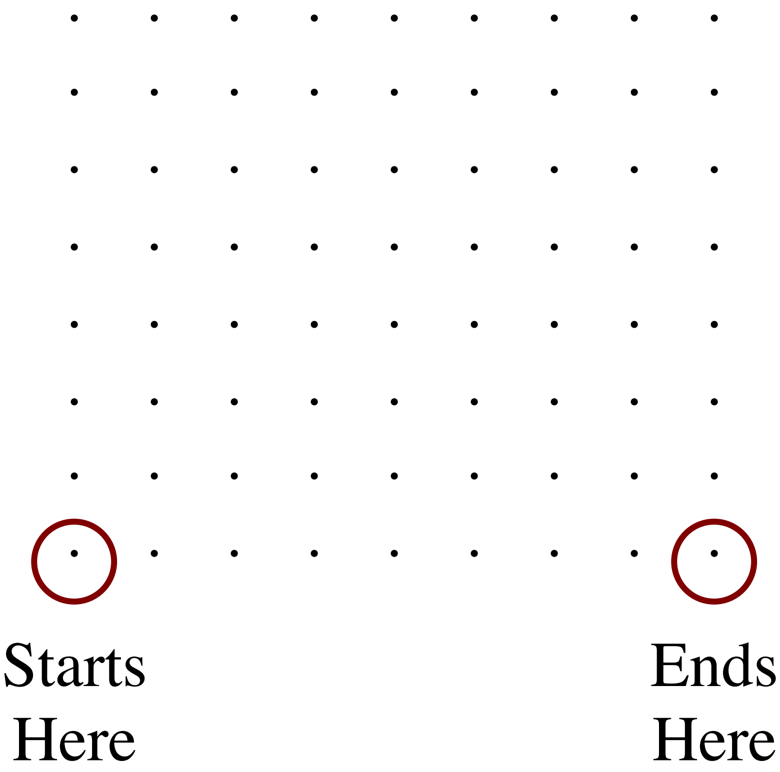
Move



Change Color



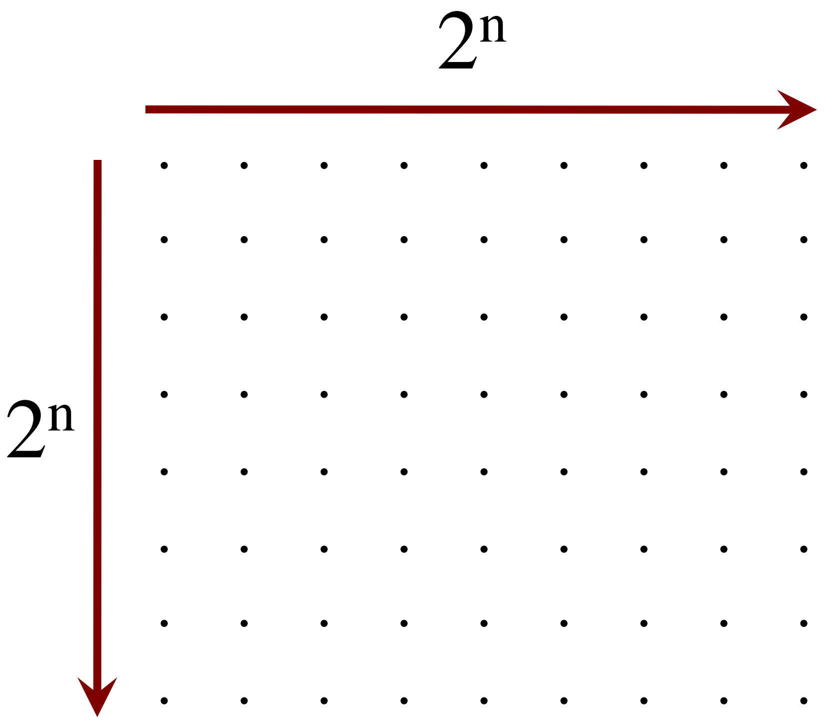
Example: Space Filling Curves



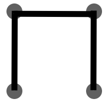
Challenge

- Draw a curve that
    - Starts in the left corner
    - Ends in the right corner
    - Touches every grid point
    - Does not touch or cross itself anywhere
- Useful for analysis of 2-dimensional data

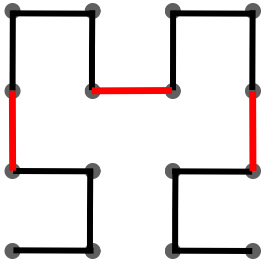
Hilbert's Space Filling Curve



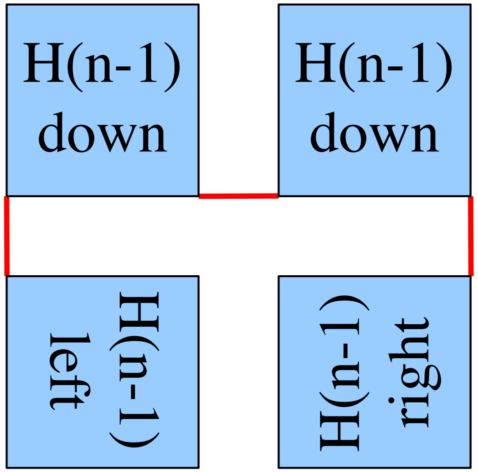
Hilbert(1):



Hilbert(2):



Hilbert(n):



H(n-1) down

H(n-1) down

H(n-1) left

H(n-1) right

Hilbert's Space Filling Curve

Basic Idea

- Given a box
- Draw $2^n \times 2^n$ grid in box
- Trace the curve
- As n goes to ∞, curve fills box