Lecture 26

Sequence Algorithms (Continued)

Announcements for This Lecture

Lab/Finals

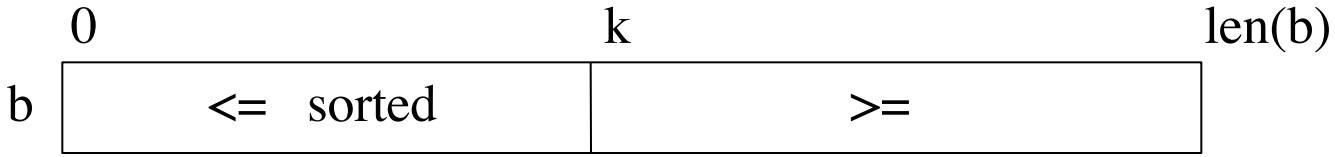
- **Lab 12** is the final lab
  - Can use Consulting hours
  - Due **next Wednesday 9:30**

- **Final: Dec 10th 2:00-4:30pm**
  - Study guide is posted
  - Announce reviews next week.
- **Conflict with Final time?**
  - Submit to conflict to CMS **by next TUESDAY!**

Assignments

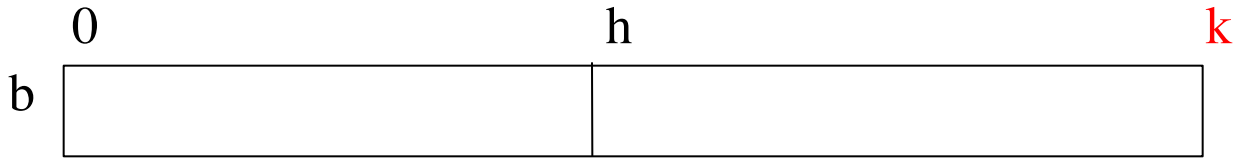
- **A6** is now graded
  - **Mean**: 91.5 **Median**: 95
  - **Std Dev**: 12.9
  - **Mean**: 11 hr **Median**: 10 hr
  - **Std Dev**: 5.7 hr
  - **SEVERAL** AI hearings
- **A7** is due **Tuesday Dec. 4**
  - Due at midnight
  - Some extensions possible

Recall: Horizontal Notation



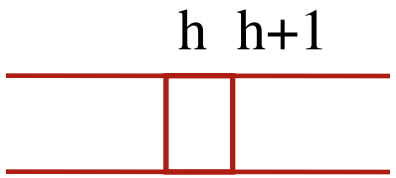
Example of an assertion about an sequence b. It asserts that:

1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]



Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.

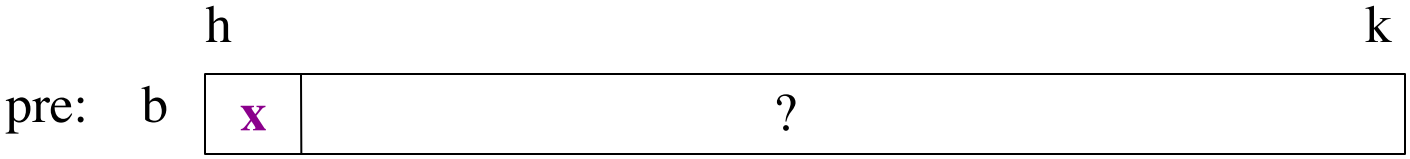
b[h .. k – 1] has k – h elements in it.



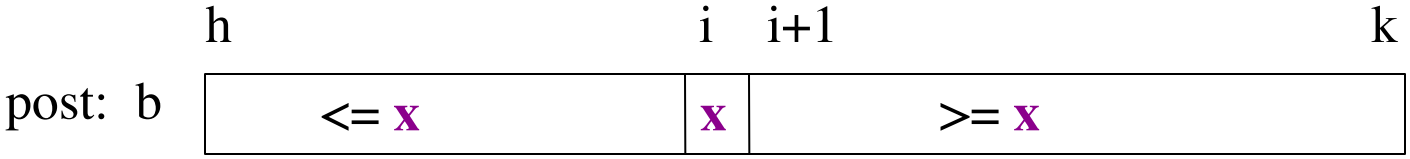
(h+1) – h = 1

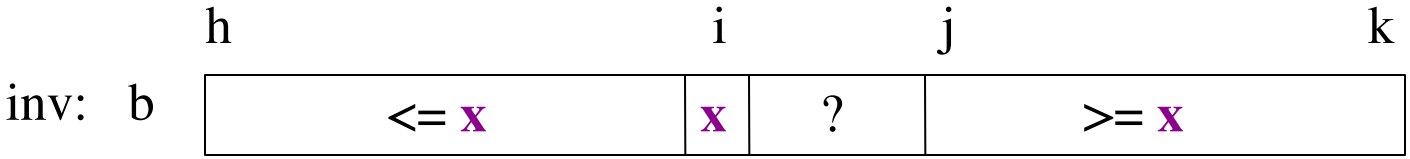
Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:



- Swap elements of b[h..k] and store in j to truthify post:





- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

partition(b,h,k), not partition(b[h:k+1])

Remember, slicing always copies the list!

We want to partition the **original** list

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
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        if b[i+1] >= x:
            # Move to end of block.
            swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

<= x		x	?			>= x		
h		i	i+1			j		k
1	2	3	1	5	0	6	3	8

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
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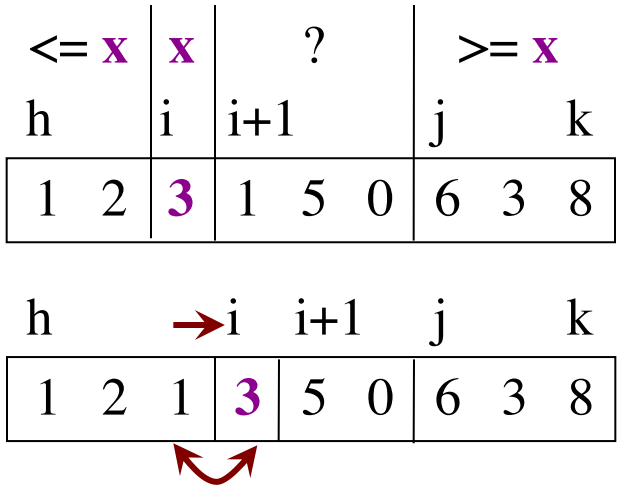

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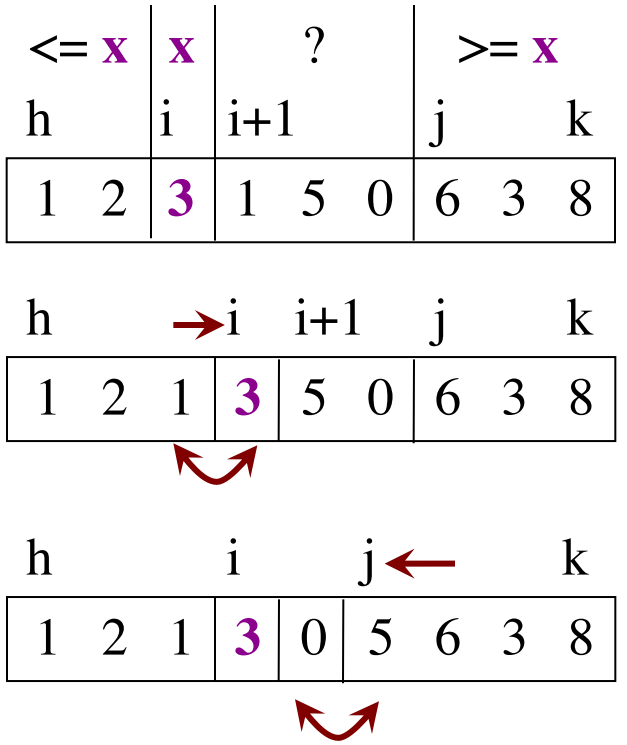

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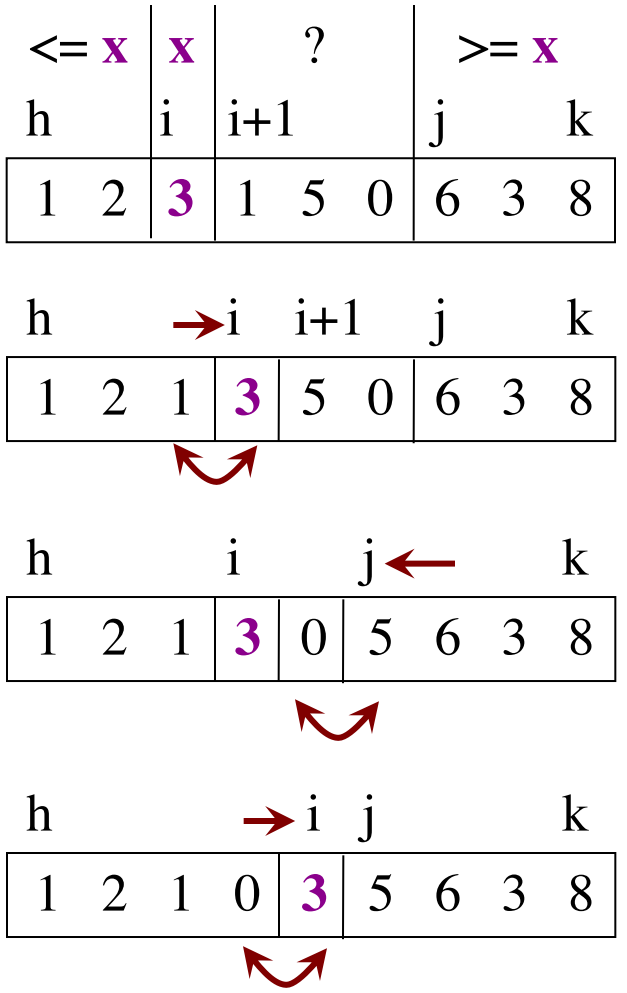
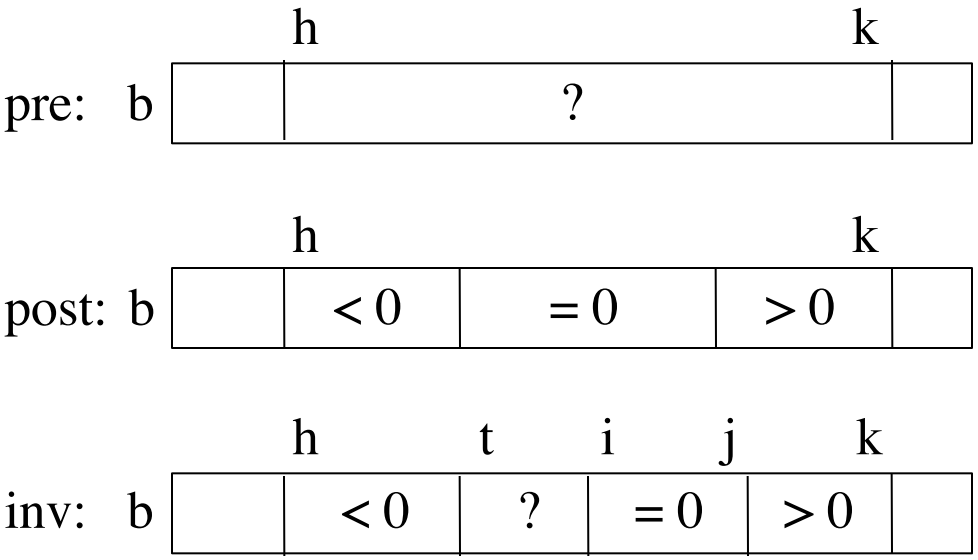


Dutch National Flag Variant

- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all



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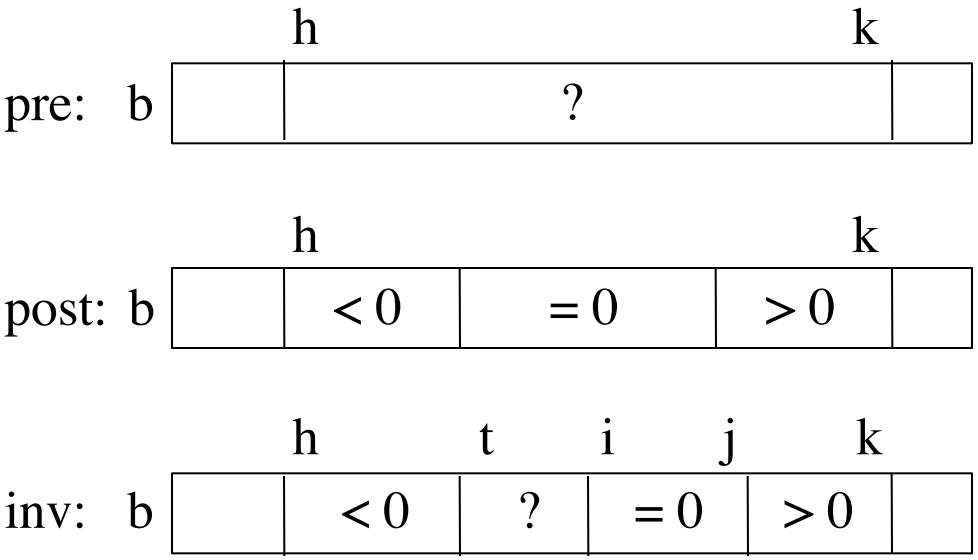

pre: t = h,
i = k+1,
j = k
post: t = i

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		?			= 0		> 0	
h		t			i	j		k
-1	-2	3	-1	0	0	0	6	3

Dutch National Flag Algorithm

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        elif b[i-1] == 0:
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        else:
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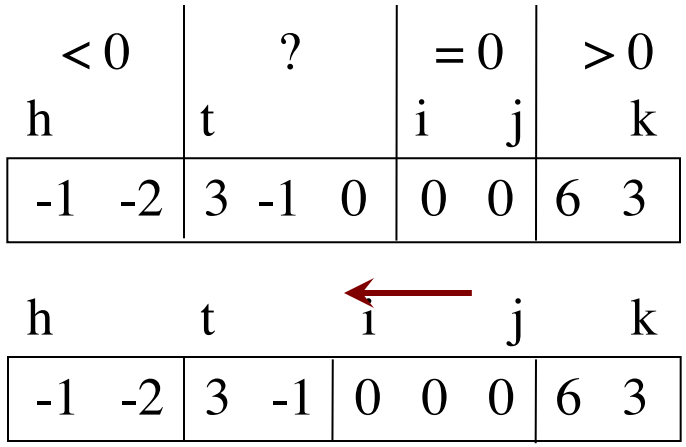

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            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
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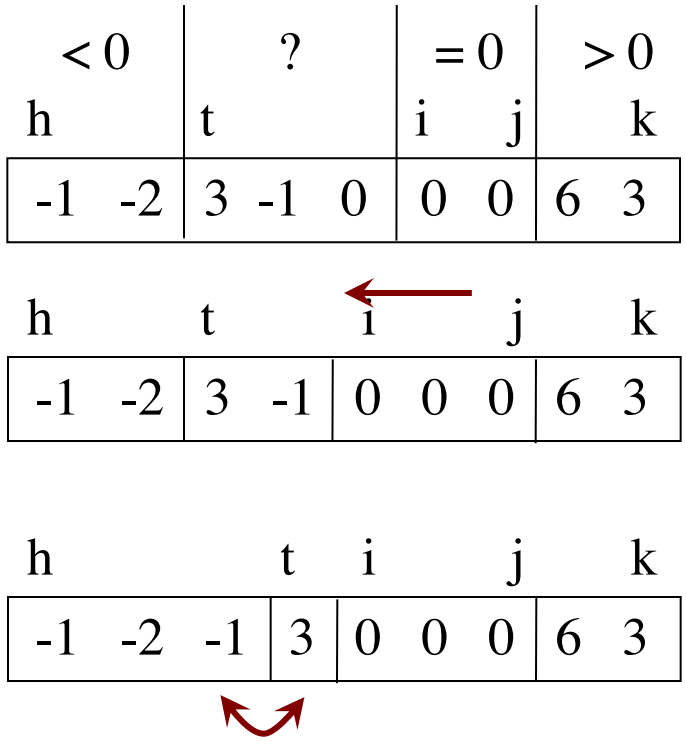
```



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    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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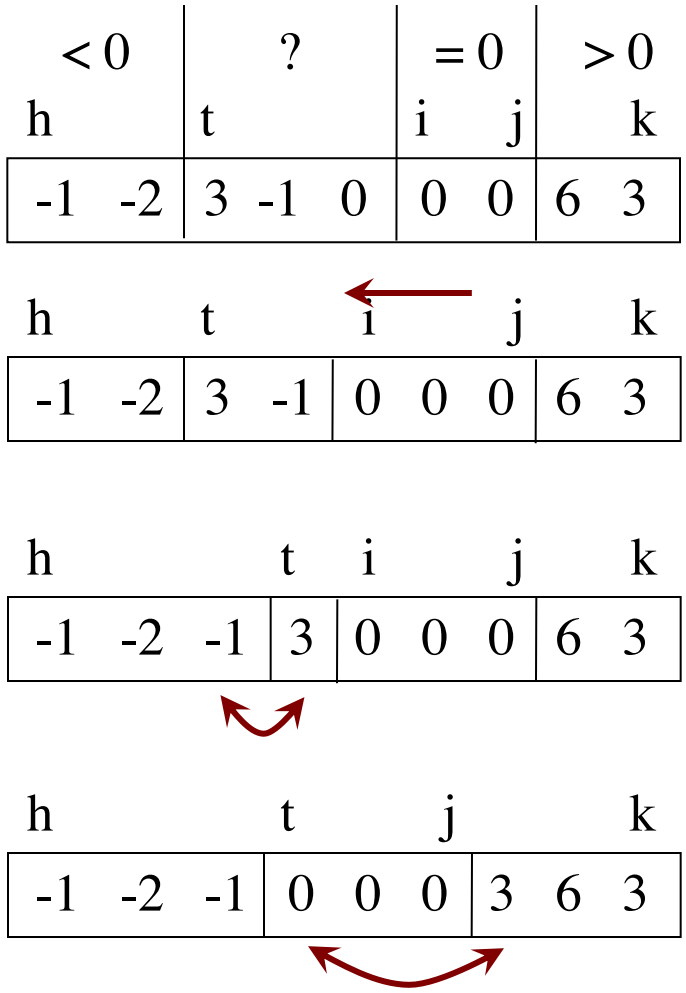
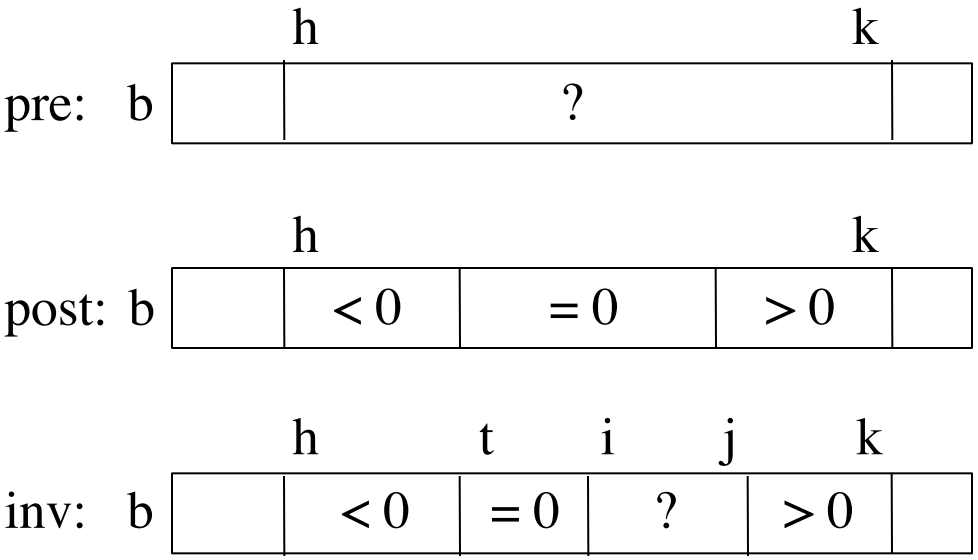
```



Changing the Invariant

- Different invariants = different code
  - Need to change how we initialize, stop
  - Also need to change the body of the loop



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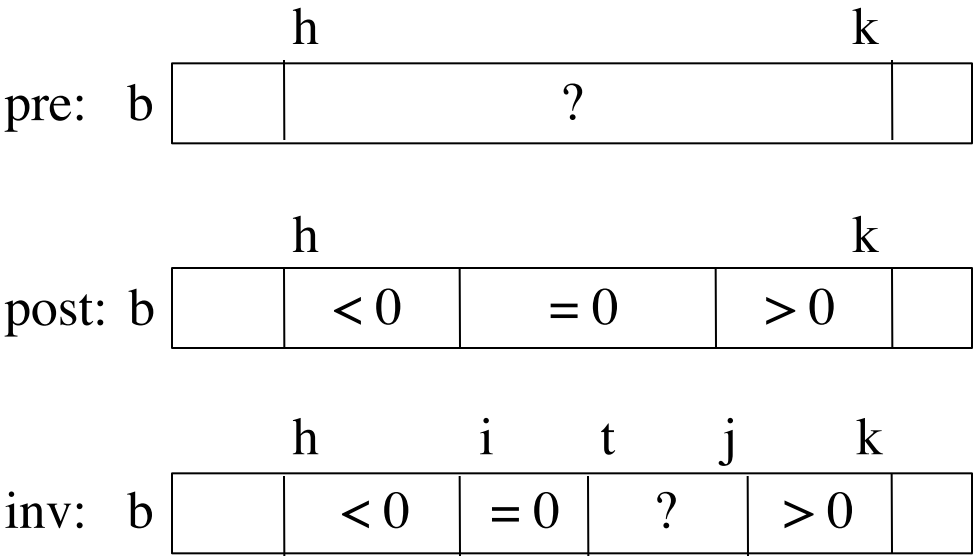

pre: t = h,
i = h,
j = k
post: **t = j+1**

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[???] < 0:
            ???

        elif b[???] == 0:
            ???
        else:
            ???

    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		= 0		?			> 0	
h		i		t		j		k
-1	-2	0	0	3	-1	0	6	3

Changing the Invariant

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    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            ???
        else:
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    return (i, j)
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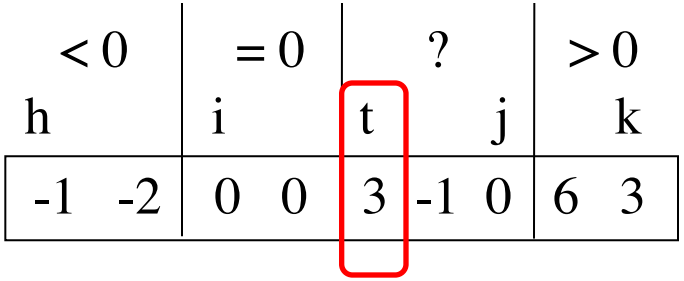

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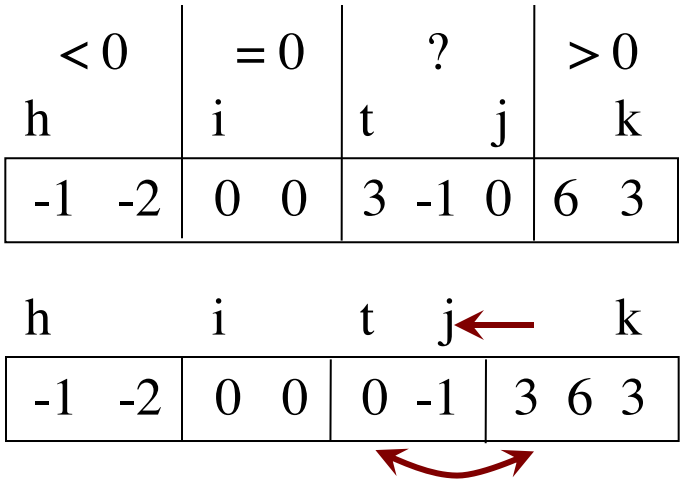
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            ???
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



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    while t < j+1:
        if b[t] < 0:
            ???
        elif b[t] == 0:
            t = t+1
        else:
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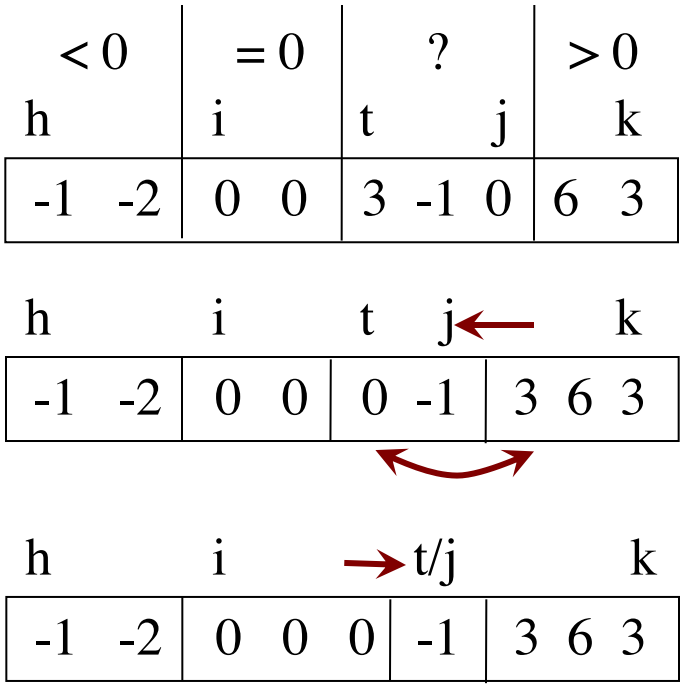

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    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
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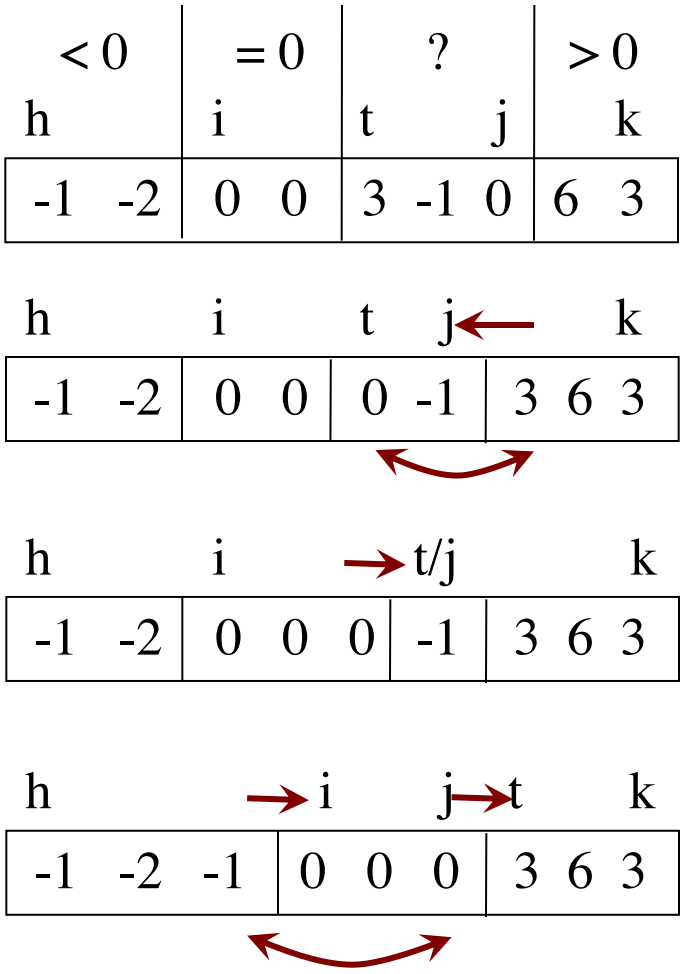

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    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # b[h..i-1] <, b[i..j] =, b[j+1..k] >
    return (i, j)
```

VS

```
def dnf(b, h, k):
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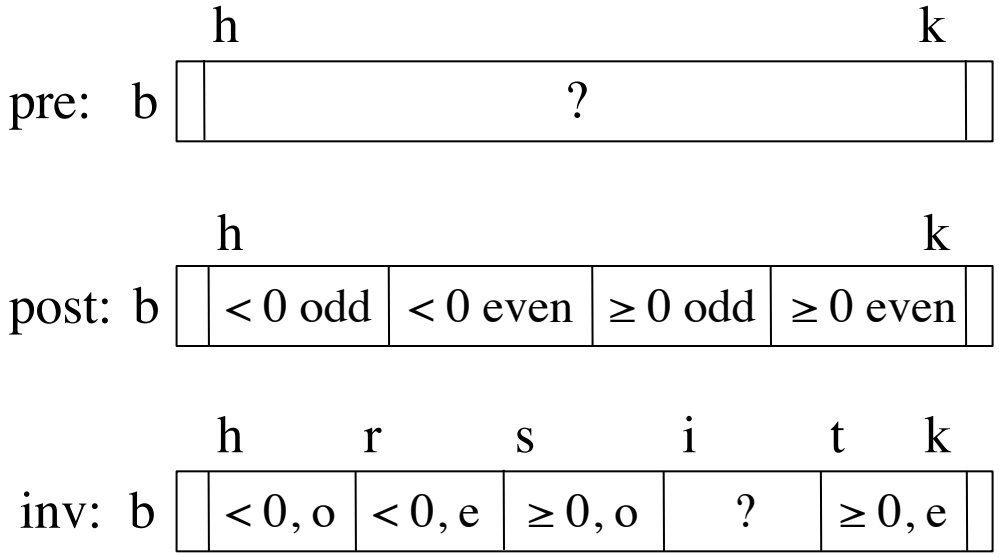
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # b[h..i-1] <, b[i..j] =, b[j+1..k] >
    return (i, j)
```

Flag of Mauritius

- Now we have four colors!
  - Negatives: ‘red’ = odd, ‘purple’ = even
  - Positives: ‘yellow’ = odd, ‘green’ = even



Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h		r		s		i				t	k
-1	-3	-5	-4	7	5	-2	-6	1	0	2	4

One swap is not good enough

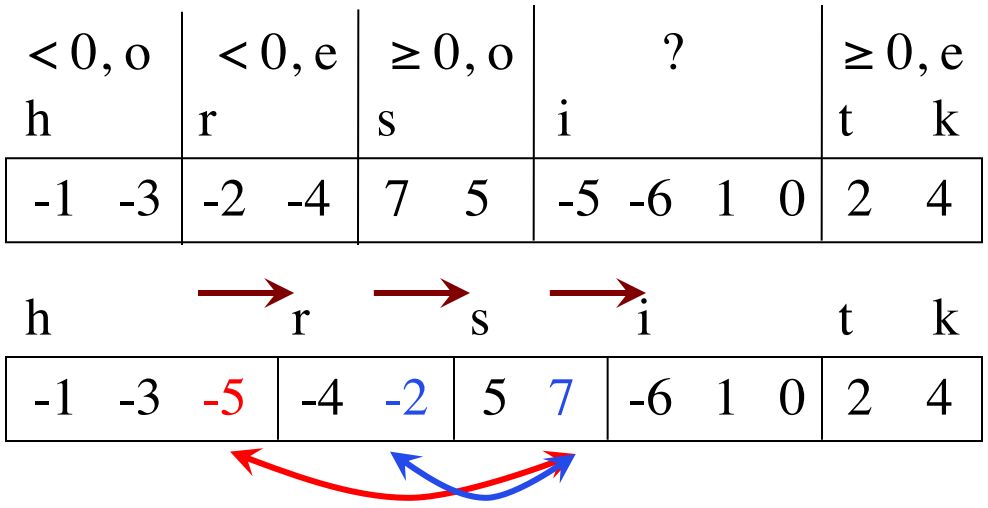
Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3	-5 -4	-2 5	7 -6 1 0	2 4

Need two swaps for two spaces

Flag of Mauritius



And adjust the loop variables

Flag of Mauritius

< 0, o			< 0, e			?			≥ 0, e	
h			r=s			i			t	k
-1	-3	-7	-4	-2	-6	-5	1	0	2	4

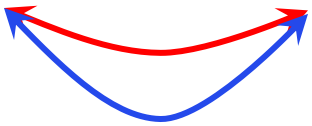
h			r=s			i			t	k
-1	-3	-7	-5	-2	-6	-4	1	0	2	4

BUT NOT ALWAYS!

Flag of Mauritius

< 0, o			< 0, e			?			≥ 0, e	
h			r=s			i			t	k
-1	-3	-7	-4	-2	-6	-5	1	0	2	4

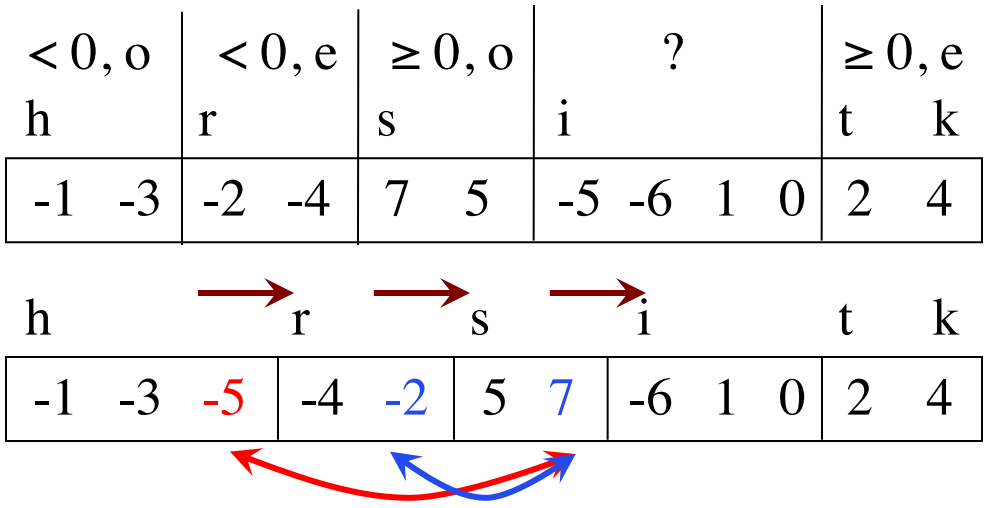
h			r=s			i			t	k
-1	-3	-7	-4	-2	-6	-5	1	0	2	4



BUT NOT ALWAYS!

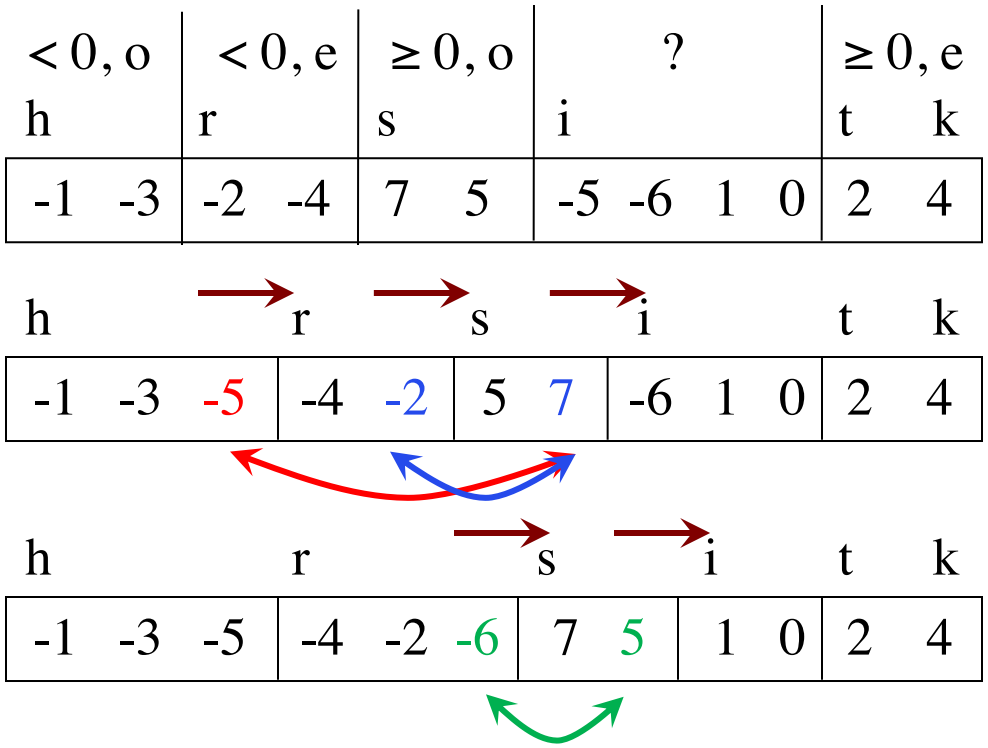
Have to check if second swap is okay

Flag of Mauritius



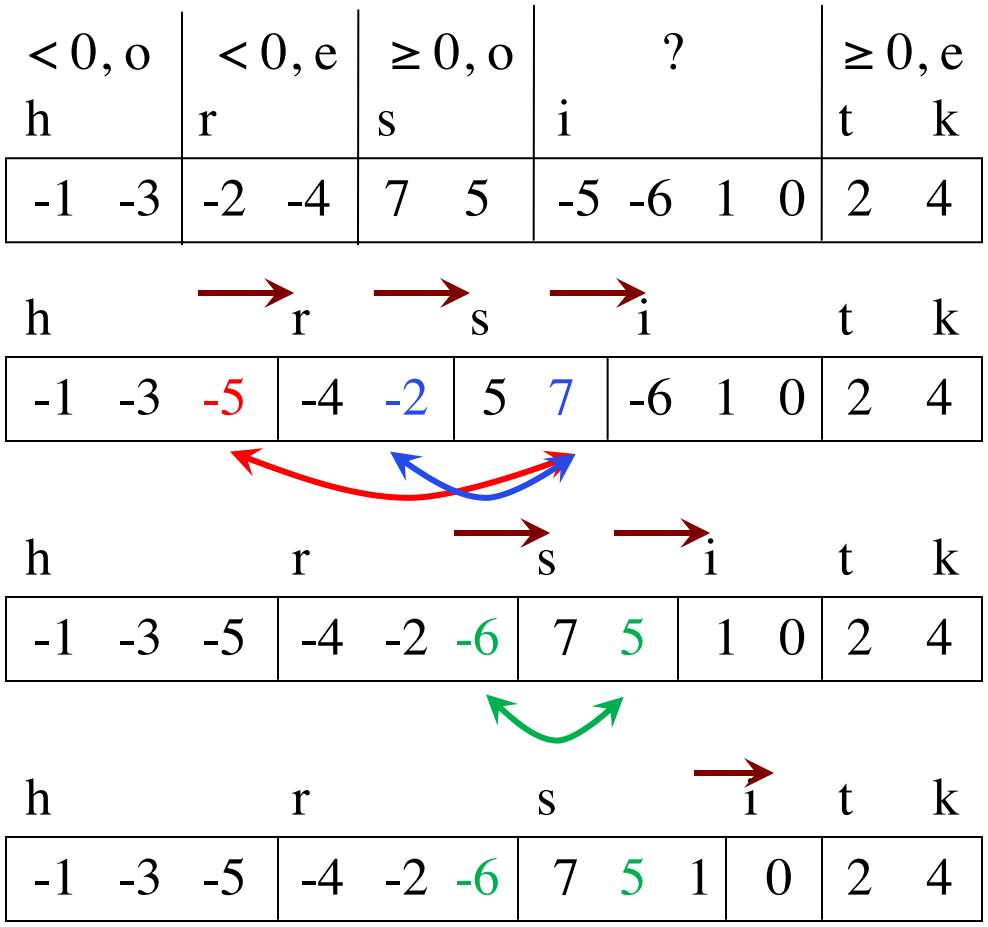
See algorithms.py for Python code

Flag of Mauritius



See algorithms.py for Python code

Flag of Mauritius



See algorithms.py for Python code

Extras Not Covered in Class

Loaded Dice

- Sequence p of length n represents n-sided die
  - Contents of p sum to 1
  - p[k] is probability die rolls the number k

1	2	3	4	5	6
0.1	0.1	0.1	0.1	0.3	0.3

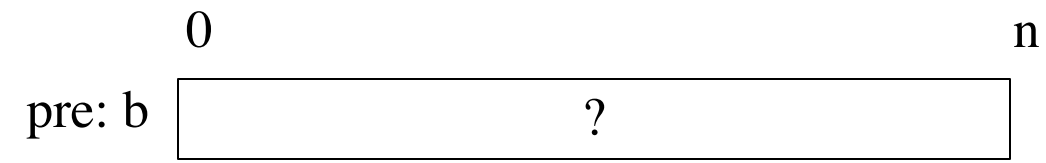
weighted d6, favoring 5, 6

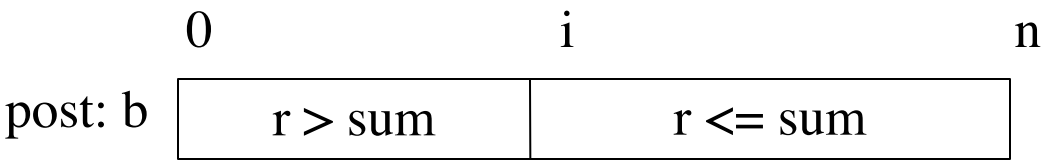
- Goal: Want to "roll the die"
  - Generate random number r between 0 and 1
  - Pick p[i] such that $p[i-1] < r \leq p[i]$

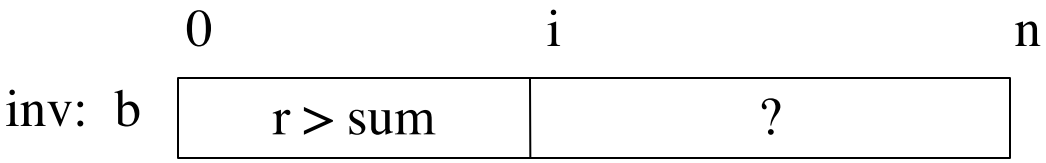
0.1	0.1	0.1	0.1	0.3	0.3
0.1	0.2	0.3	0.4	0.7	1.0

Loaded Dice

- **Want**: Value i such that p[i-1] < r <= p[i]







- Same as precondition if i = 0
- Postcondition is invariant + false loop condition

Loaded Dice

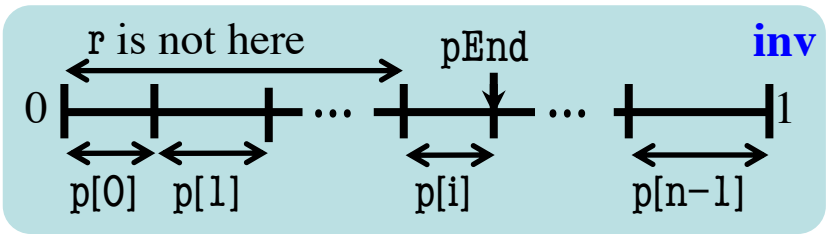
```
def roll(p):
    """Returns: randint in 0..len(p)-1; i returned with prob. p[i]
    Precondition: p list of positive floats that sum to 1."""
    r = random.random()     # r in [0,1)
    # Think of interval [0,1] divided into segments of size p[i]
    # Store into i the segment number in which r falls.
    i = 0;    sum_of = p[0]
    # inv: r >= sum of p[0] .. p[i-1]; pEnd = sum of p[0] .. p[i]
    while r >= sum_of:
        sum_of = sum_of + p[i+1]
        i = i + 1

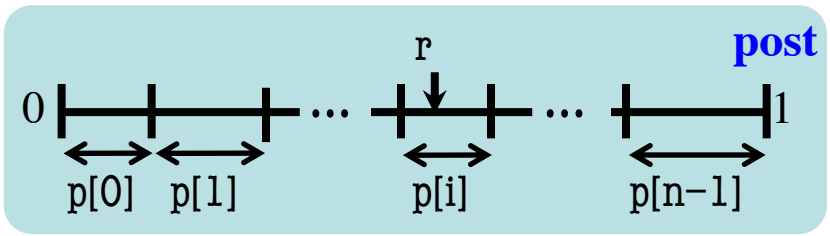
    # post: sum of p[0] .. p[i-1] <= r < sum of p[0] .. p[i]
    return i
```

Analyzing the Loop

1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep **inv** true?

r < sum





Reversing a Sequence