Lecture 26

Sequence Algorithms (Continued)

Announcements for This Lecture

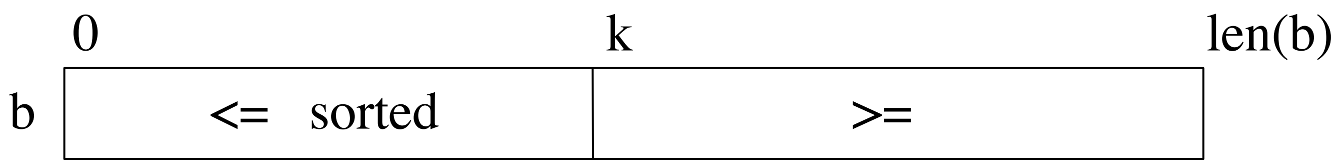
Assignment & Lab

- A6 is not graded yet
  - Done early next week
- A7 due **Sun, Dec. 3**
  - But extensions possible
  - Just ask for one!
  - But make good effort
- Lab Today: Office Hours
  - Get help on A7 aliens
  - Anyone can go to any lab

Next Week

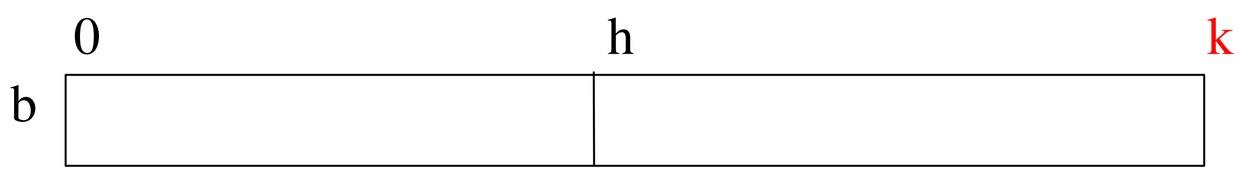
- Last Week of Class!
  - Finish sorting algorithms
  - Special final lecture
- Lab held, but is optional
  - Can use lab as extra credit
  - Also use lab time on A7
- Details about the exam
  - Multiple review sessions

Recall: Horizontal Notation



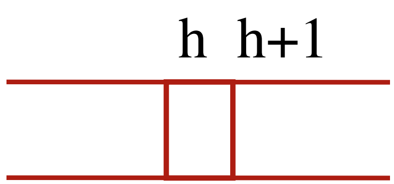
Example of an assertion about an sequence b. It asserts that:

1. b[0..k–1] is sorted (i.e. its values are in ascending order)
2. Everything in b[0..k–1] is ≤ everything in b[k..len(b)–1]



Given index h of the first element of a segment and index k of the element that follows that segment, the number of values in the segment is k – h.

b[h .. k – 1] has k – h elements in it.



(h+1) – h = 1

Partition Algorithm

- Given a sequence b[h..k] with some value x in b[h]:

	h		k
pre: b	**x**	?	

- Swap elements of b[h..k] and store in j to truthify post:

	h	i	i+1	k
post: b	<= **x**	**x**	>= **x**	

	h	i		j	k
inv: b	<= **x**	**x**	?	>= **x**	

- Agrees with precondition when i = h, j = k+1
- Agrees with postcondition when j = i+1

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

partition(b,h,k), not partition(b[h:k+1])

Remember, slicing always copies the list!

We want to partition the **original** list

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
    # invariant: b[h..i-1] < x, b[i] = x, b[j..k] >= x
    while i < j-1:
        if b[i+1] >= x:
            # Move to end of block.
            swap(b,i+1,j-1)
            j = j - 1
        else:   # b[i+1] < x
            swap(b,i,i+1)
            i = i + 1
    # post: b[h..i-1] < x, b[i] is x, and b[i+1..k] >= x
    return i
```

<= x	x	?	>= x
h	i	i+1	j k
1 2	3	1 5 0	6 3 8

Partition Algorithm Implementation

```
def partition(b, h, k):
    """Partition list b[h..k] around a pivot x = b[h]"""
    i = h; j = k+1; x = b[h]
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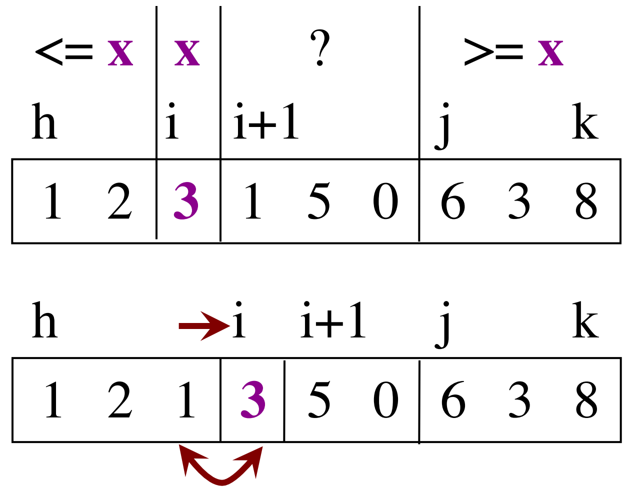

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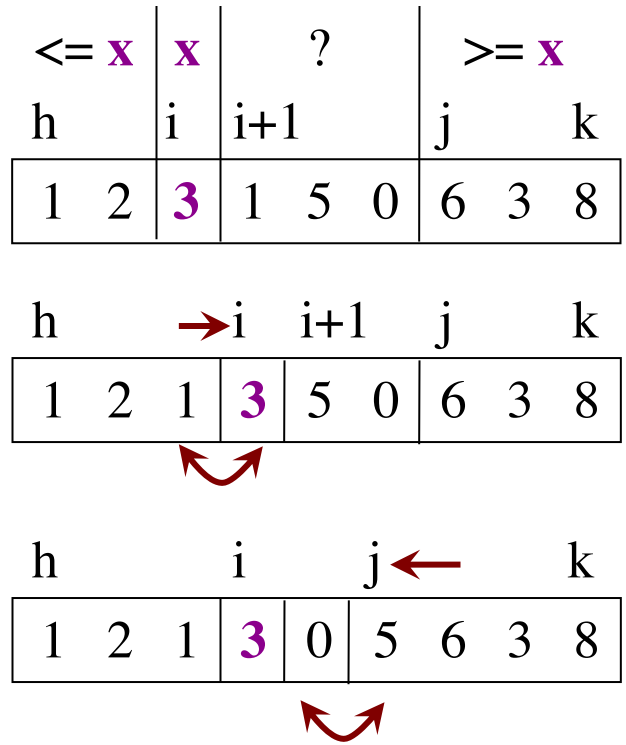

Partition Algorithm Implementation

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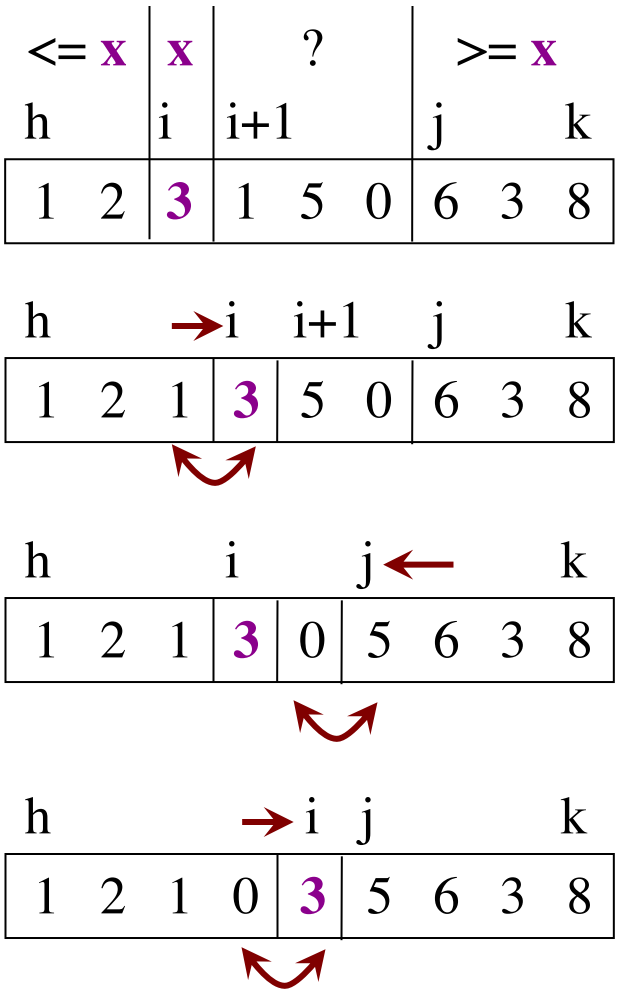
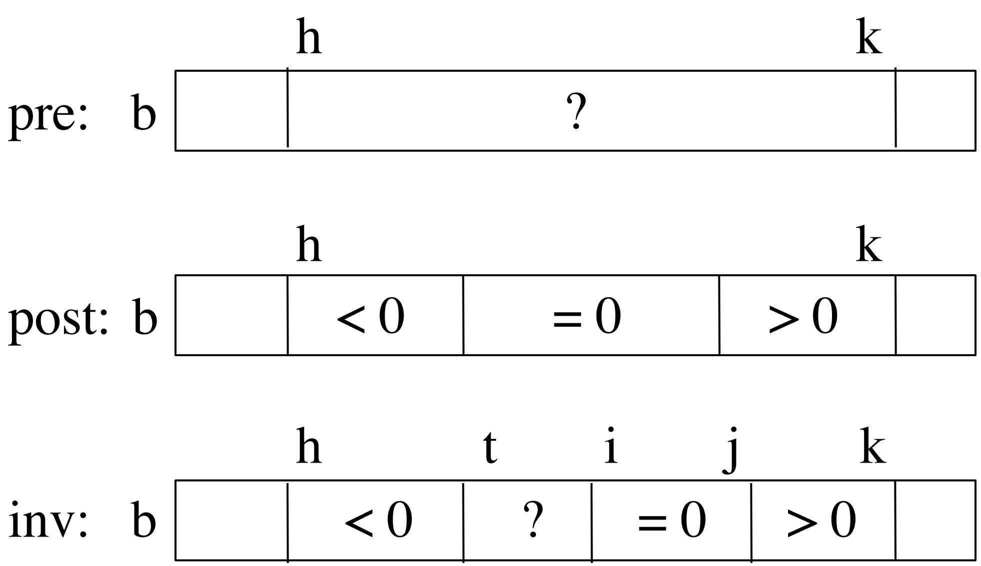


Dutch National Flag Variant

- Sequence of integer values
  - ‘red’ = negatives, ‘white’ = 0, ‘blues’ = positive
  - Only rearrange part of the list, not all



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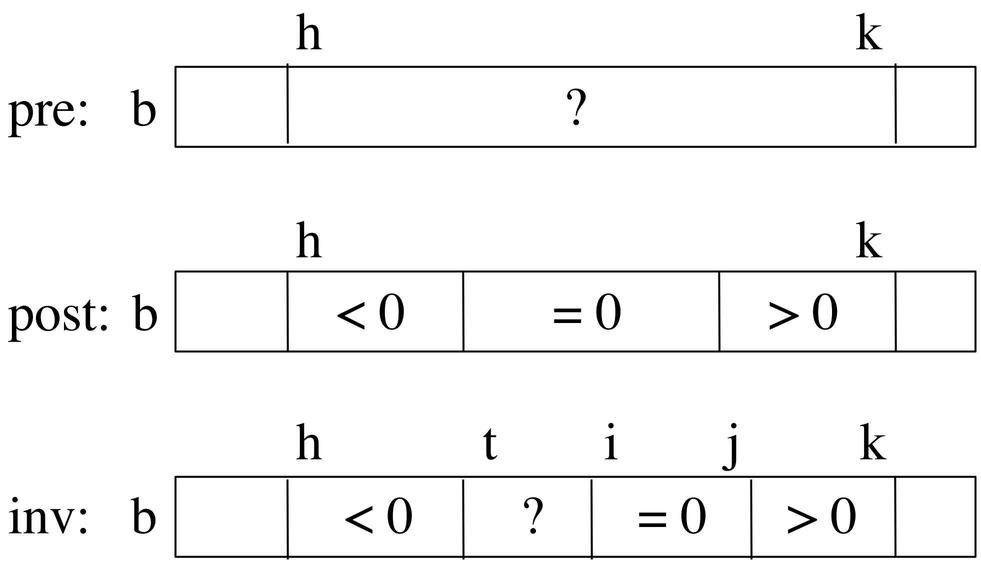

pre: t = h,
i = k+1,
j = k
post: t = i

Dutch National Flag Algorithm

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = k+1, j = k;
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0	?	= 0	> 0
h	t	i j	k
-1 -2	3 -1 0	0 0	6 3

Dutch National Flag Algorithm

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def dnf(b, h, k):
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    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		?			= 0		> 0	
h		t			i	j		k
-1	-2	3	-1	0	0	0	6	3

h		t		i		j		k
-1	-2	3	-1	0	0	0	6	3

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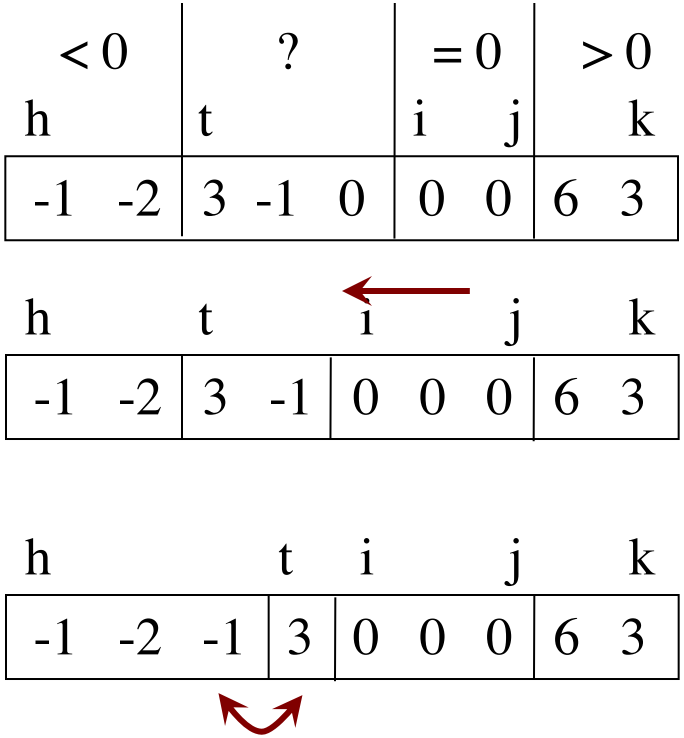
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



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```
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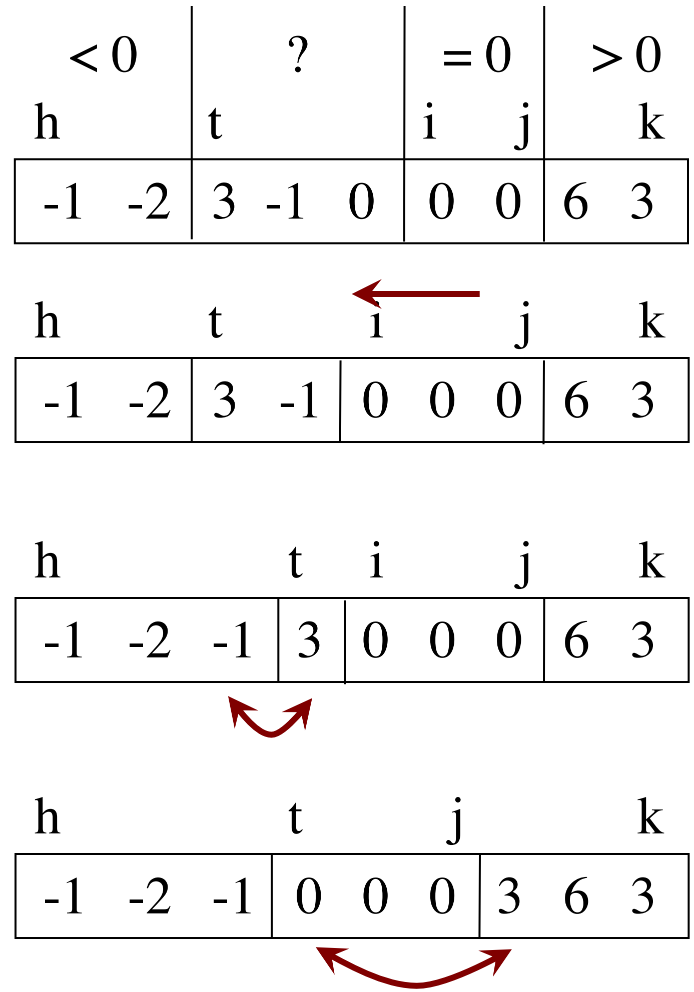
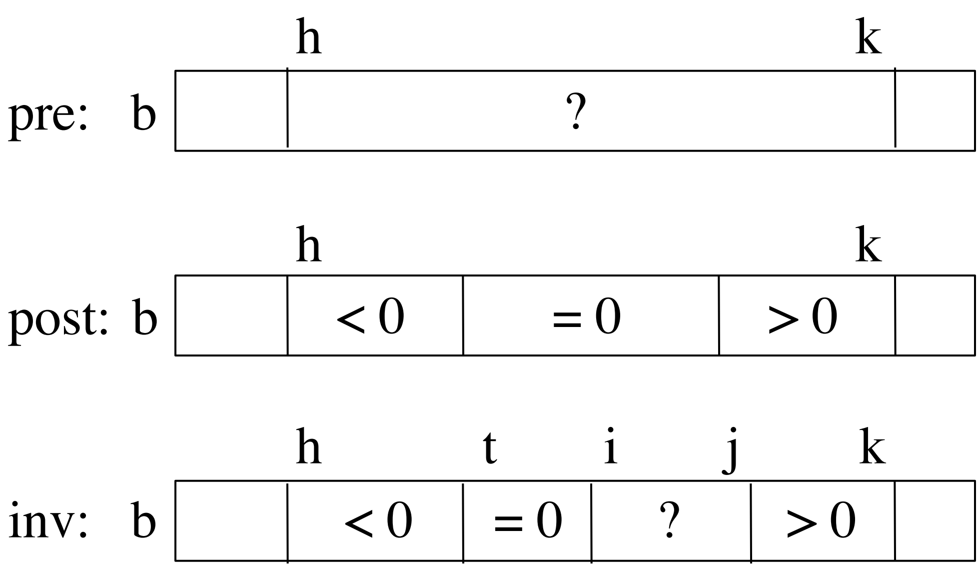
    # inv: b[h..t-1] < 0, b[t..i-1] ?, b[i..j] = 0, b[j+1..k] > 0
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```



Changing the Invariant

- Different invariants = different code
  - Need to change how we initialize, stop
  - Also need to change the body of the loop



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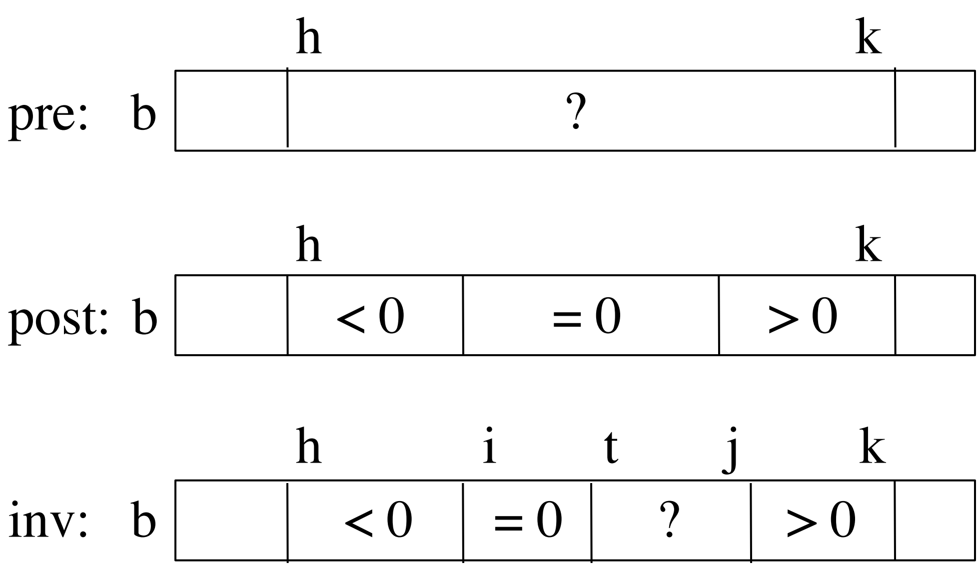

pre: t = h,
i = h,
j = k
post: **t = j+1**

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[???] < 0:
            ???

        elif b[???] == 0:
            ???
        else:
            ???

    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0	= 0	?	> 0
h	i	t j	k
-1 -2	0 0	3 -1 0	6 3

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            ???
        else:
            ???

    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0		= 0		?			> 0	
h		i		t		j		k
-1	-2	0	0	3	-1	0	6	3

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            ???
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0	= 0	?	> 0
h	i	t j	k
-1 -2	0 0	3 -1 0	6 3

h	i	t j	k
-1 -2	0 0	0 -1	3 6 3

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            ???

        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
    return (i, j)
```

< 0	= 0	?	> 0
h	i	t j	k
-1 -2	0 0	3 -1 0	6 3

h	i	t j	k
-1 -2	0 0	0 -1	3 6 3

h	i	t/j	k
-1 -2	0 0 0	-1	3 6 3

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points as a tuple (i,j)"""
    t = h; i = h, j = k;
    # inv: b[h..t-1] < 0, b[i..t-1] = 0, b[t..j] ?, b[j+1..k] > 0
    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # post: b[h..i-1] < 0, b[i..j] = 0, b[j+1..k] > 0
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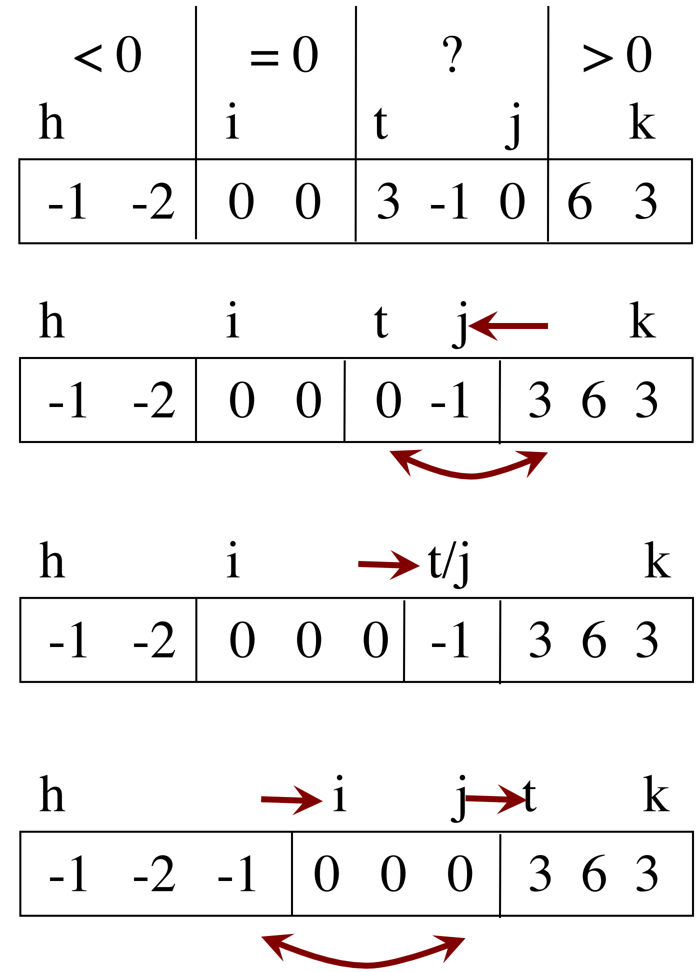

Changing the Invariant

```
def dnf(b, h, k):
    """Returns: partition points"""
    t = h; i = h, j = k;
    # b[h..t-1] <, b[i..t-1] =, b[t..j] ?, b[j+1..k] >
    while t < j+1:
        if b[t] < 0:
            swap(b,t,i)
            i = i+1; t = t+1;
        elif b[t] == 0:
            t = t+1
        else:
            swap(b,t,j)
            j = j-1
    # b[h..i-1] <, b[i..j] =, b[j+1..k] >
    return (i, j)
```

VS

```
def dnf(b, h, k):
    """Returns: partition points"""
    t = h; i = k+1, j = k;
    # b[h..t-1] <, b[t..i-1] ?, b[i..j] =, b[j+1..k] >
    while t < i:
        if b[i-1] < 0:
            swap(b,i-1,t)
            t = t+1
        elif b[i-1] == 0:
            i = i-1
        else:
            swap(b,i-1,j)
            i = i-1; j = j-1
    # b[h..i-1] <, b[i..j] =, b[j+1..k] >
    return (i, j)
```

Flag of Mauritius

- Now we have four colors!
  - Negatives: ‘red’ = odd, ‘purple’ = even
  - Positives: ‘yellow’ = odd, ‘green’ = even

		h		k	
pre:	b		?		

		h				k	
post:	b		< 0 odd	< 0 even	≥ 0 odd	≥ 0 even	

		h	r	s	i	t	k
inv:	b		< 0, o	< 0, e	≥ 0, o	?	≥ 0, e

Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h		r		s		i				t	k
-1	-3	-5	-4	7	5	-2	-6	1	0	2	4

One swap is not good enough

Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h		r		s		i				t	k
-1	-3	-5	-4	-2	5	7	-6	1	0	2	4

Need two swaps for two spaces

Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2	5 7	-6 1 0	2 4

And adjust the loop variables

Flag of Mauritius

< 0, o		< 0, e		≥ 0, o		?				≥ 0, e	
h		r		s		i				t	k
-1	-3	-2	-4	7	5	-5	-6	1	0	2	4

h			r		s		i			t	k
-1	-3	-5	-4	-2	5	7	-6	1	0	2	4

h			r			s		i		t	k
-1	-3	-5	-4	-2	-6	7	5	1	0	2	4

See algorithms.py for Python code

Flag of Mauritius

< 0, o	< 0, e	≥ 0, o	?	≥ 0, e
h	r	s	i	t k
-1 -3	-2 -4	7 5	-5 -6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2	5 7	-6 1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2 -6	7 5	1 0	2 4

h	r	s	i	t k
-1 -3 -5	-4 -2 -6	7 5 1	0	2 4

See algorithms.py for Python code

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].

Linear Search

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- **Better**: Store an integer in i to truthify result condition post:

  post: 1. v is not in b[h..i-1]

  2. i = k OR v = b[i]

Linear Search

- **Vague**: Find first occurrence of v in b[h..k-1].
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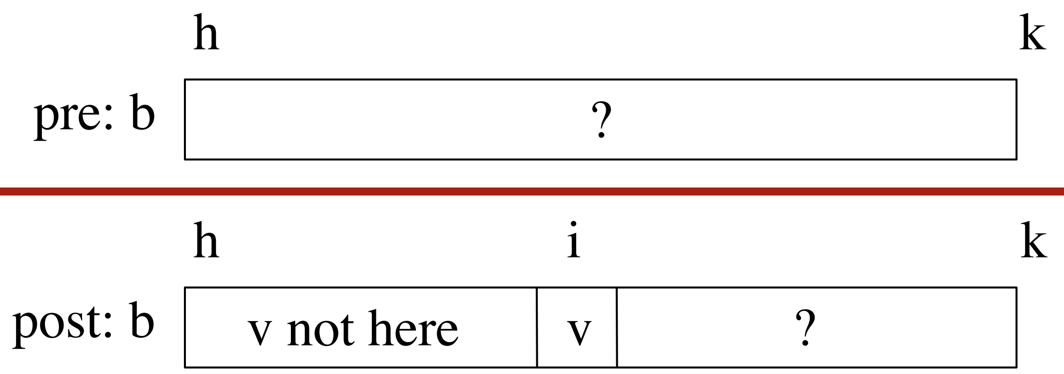

Linear Search

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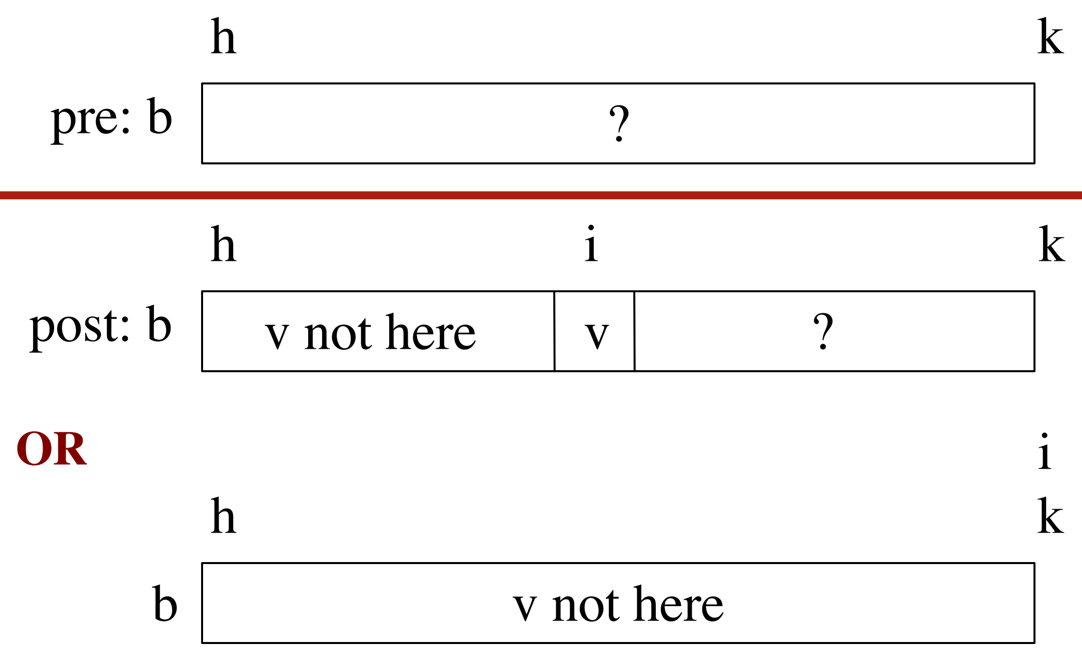

Linear Search

	h		k
pre: b	?		

	h	i		k
post: b	v not here	v	?	

OR

	h	i k
b	v not here	

	h	i	k
inv: b	v not here	?	

Linear Search

```
def linear_search(b,v,h,k):
    """Returns: first occurrence of v in b[h..k-1]"""
    # Store in i index of the first v in b[h..k-1]
    i = h

    # invariant: v is not in b[0..i-1]
    while i < k and b[i] != v:
        i = i + 1

    # post: v is not in b[h..i-1]
    #       i >= k or b[i] == v
    return i if i < k else -1
```

Analyzing the Loop

1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep the invariant **inv** true?

Binary Search

- **Vague:** Look for v in **sorted** sequence segment b[h..k].

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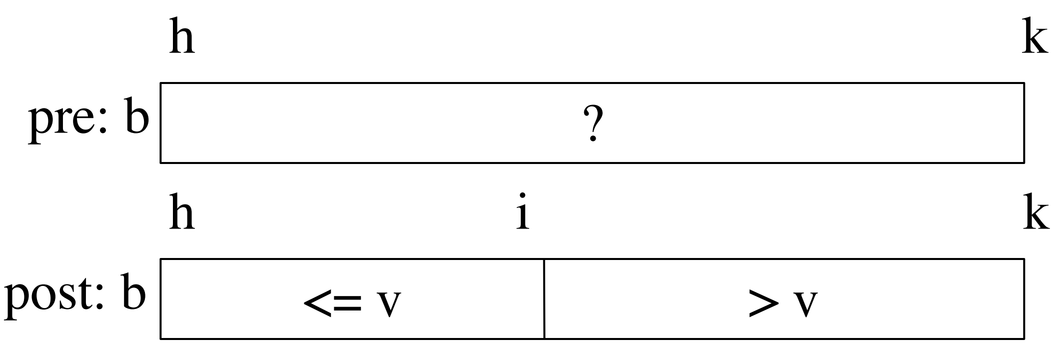
- **Better:**
  - Precondition: b[h..k-1] is sorted (in ascending order).
  - Postcondition: b[h..i] <= v and v < b[i+1..k-1]
- Below, the array is in non-descending order:



Binary Search

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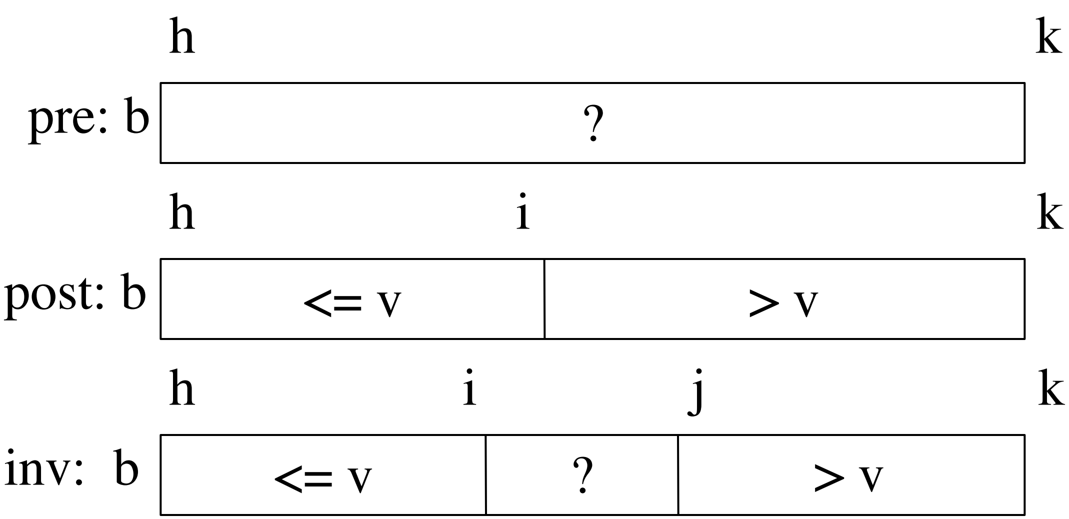
  - Postcondition: b[h..i] <= v and v < b[i+1..k-1]
- Below, the array is in non-descending order:



Called binary search because each iteration of the loop cuts the array segment still to be processed in half

Extras Not Covered in Class

Loaded Dice

- Sequence p of length n represents n-sided die
  - Contents of p sum to 1
  - p[k] is probability die rolls the number k

1	2	3	4	5	6
0.1	0.1	0.1	0.1	0.3	0.3

weighted d6, favoring 5, 6

- Goal: Want to "roll the die"
  - Generate random number r between 0 and 1
  - Pick p[i] such that p[i-1] $<$ r $\leq$ p[i]

0.1	0.1	0.1	0.1	0.3	0.3
0.1	0.2	0.3	0.4	0.7	1.0

Loaded Dice

- **Want**: Value i such that p[i-1] < r <= p[i]

	0	n
pre: b	?	

	0	i	n
post: b	r > sum	r <= sum	

	0	i	n
inv: b	r > sum	?	

- Same as precondition if i = 0
- Postcondition is invariant + false loop condition

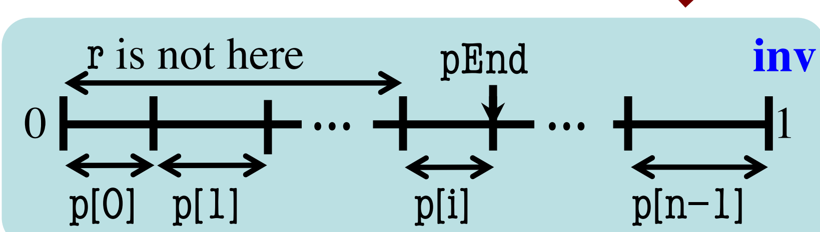
Loaded Dice

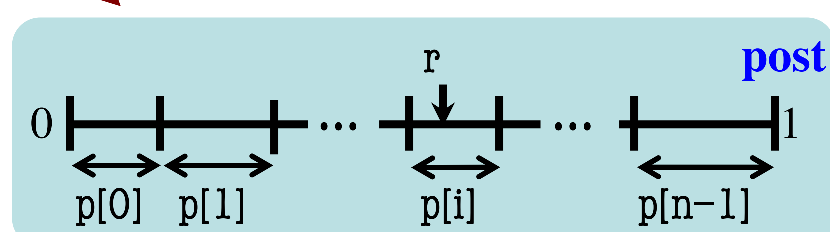
```
def roll(p):
    """Returns: randint in 0..len(p)-1; i returned with prob. p[i]
    Precondition: p list of positive floats that sum to 1."""
    r = random.random()     # r in [0,1)
    # Think of interval [0,1] divided into segments of size p[i]
    # Store into i the segment number in which r falls.
    i = 0;    sum_of = p[0]
    # inv: r >= sum of p[0] .. p[i−1]; pEnd = sum of p[0] .. p[i]
    while r >= sum_of:
        sum_of = sum_of + p[i+1]
        i = i + 1

    # post: sum of p[0] .. p[i−1] <= r < sum of p[0] .. p[i]
    return i
```

Analyzing the Loop

1. Does the initialization make **inv** true?
2. Is **post** true when **inv** is true and **condition** is false?
3. Does the repetend make progress?
4. Does the repetend keep **inv** true?

r < sum





Reversing a Sequence

		h		k
pre:	b	not reversed		

		h		k
post:	b	reversed		

		h		k
change:	b	1 2 3 4 5 6 7 8 9 9 9 9		

		h		k
into	b	9 9 9 9 8 7 6 5 4 3 2 1		

		h	i	j	k
inv:	b	swapped	not reversed	swapped	