27. Means and Medians

Three Instructive Problems:

The Apportionment Problem

The Polygon Averaging Problem

The Median Filtering Problem

What?

The Apportionment Problem

How to fairly distribute 435 Congressional Districts among the 50 states.

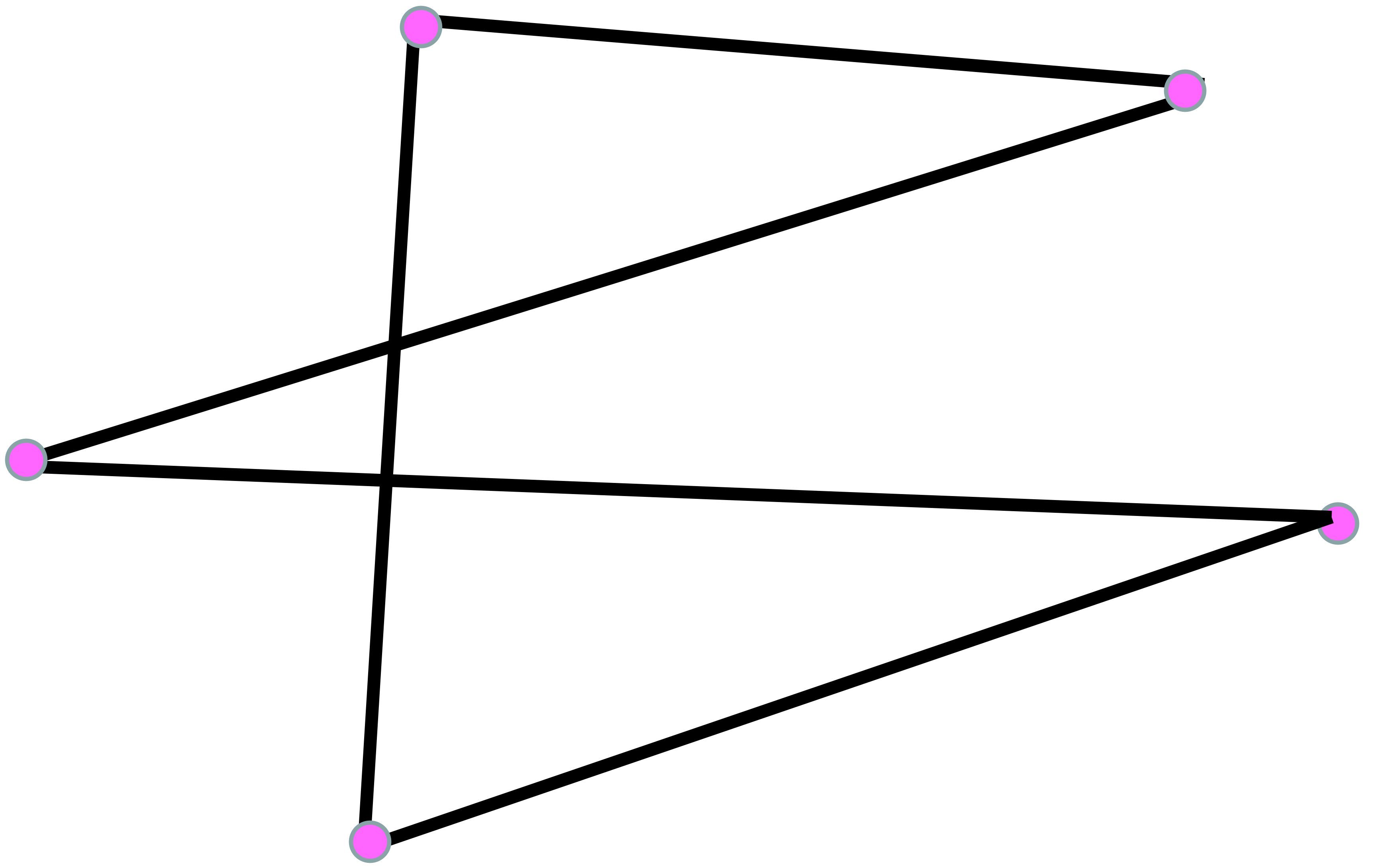
State	Pop	nDist	Pop/nDist
Alabama	4802982	7	686140
Alaska	721523	1	721523
Arizona	6412700	9	712522
Arkansas	2926229	4	731557
California	37541989	53	708339
Colorado	5044930	7	720704
Connecticut	3581628	5	716325
Delaware	900877	1	900877
Florida	18900773	27	700028
	etc		

What?

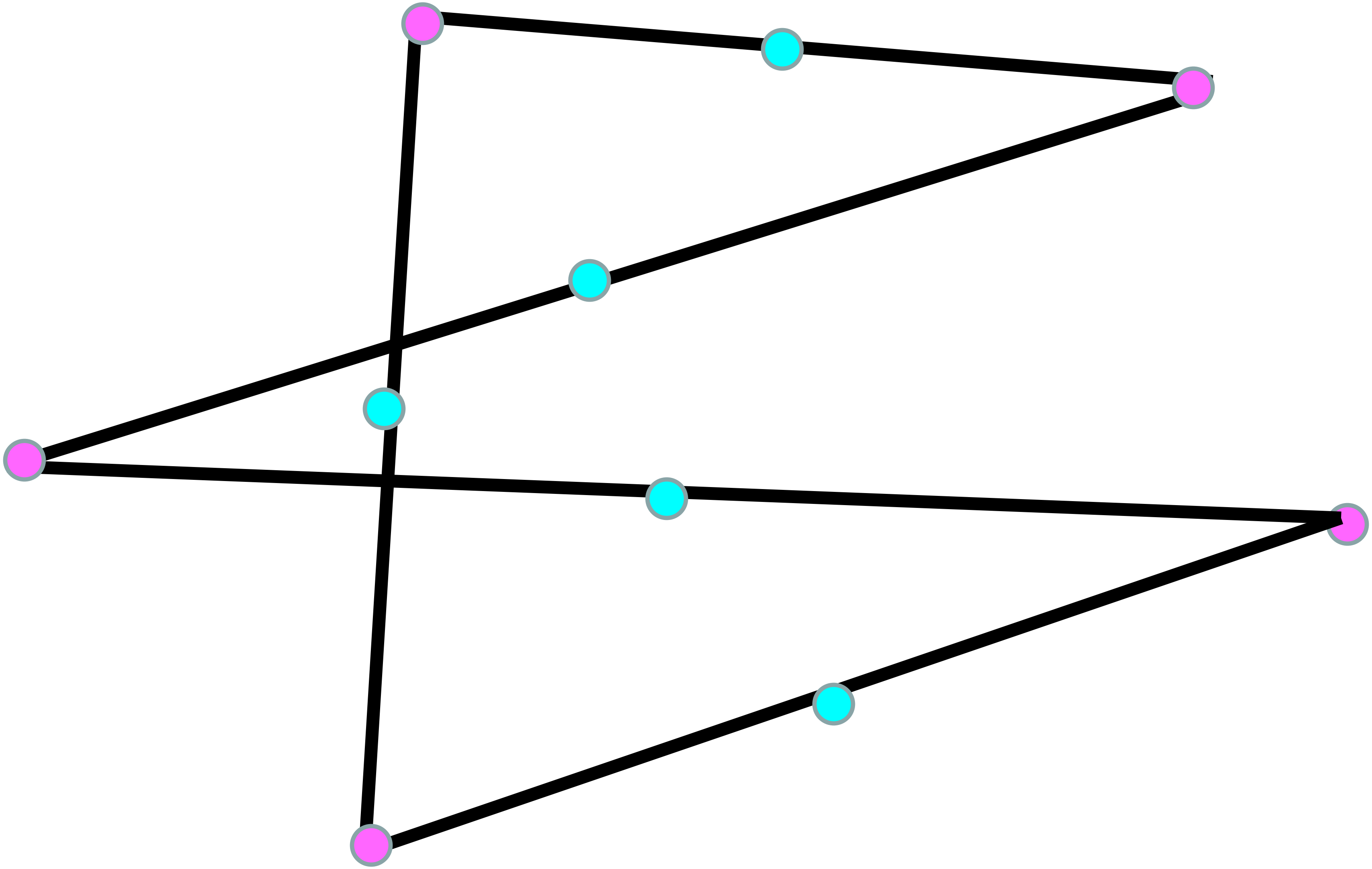
The Polygon Averaging Problem

Given a polygon, connect the midpoints of the sides. This gives a new polygon. Repeat many times.

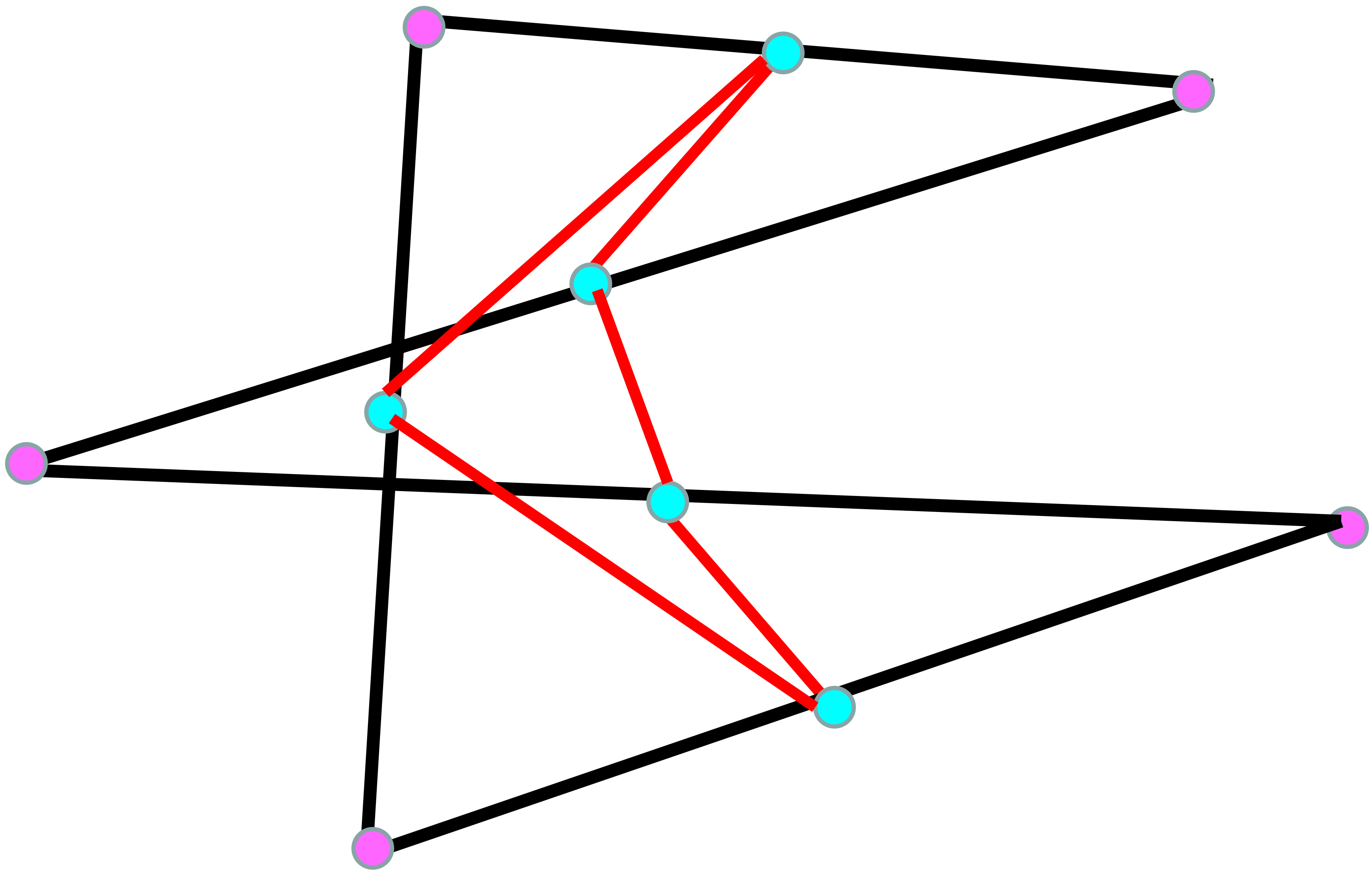
A Random Pentagon



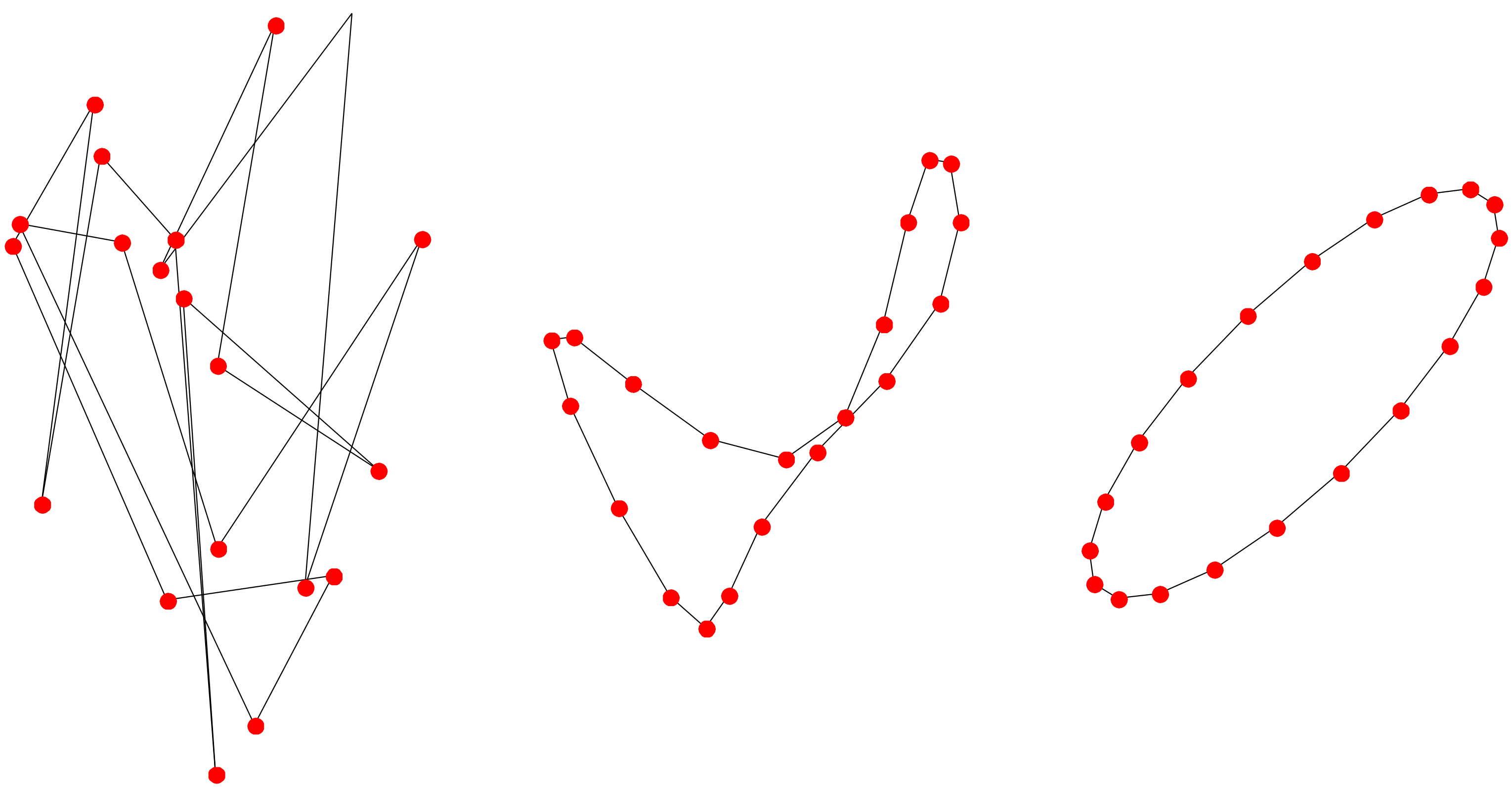
The Side Midpoints



Connect the Midpoints



The Polygon Untangles Itself and Heads Towards an Ellipse

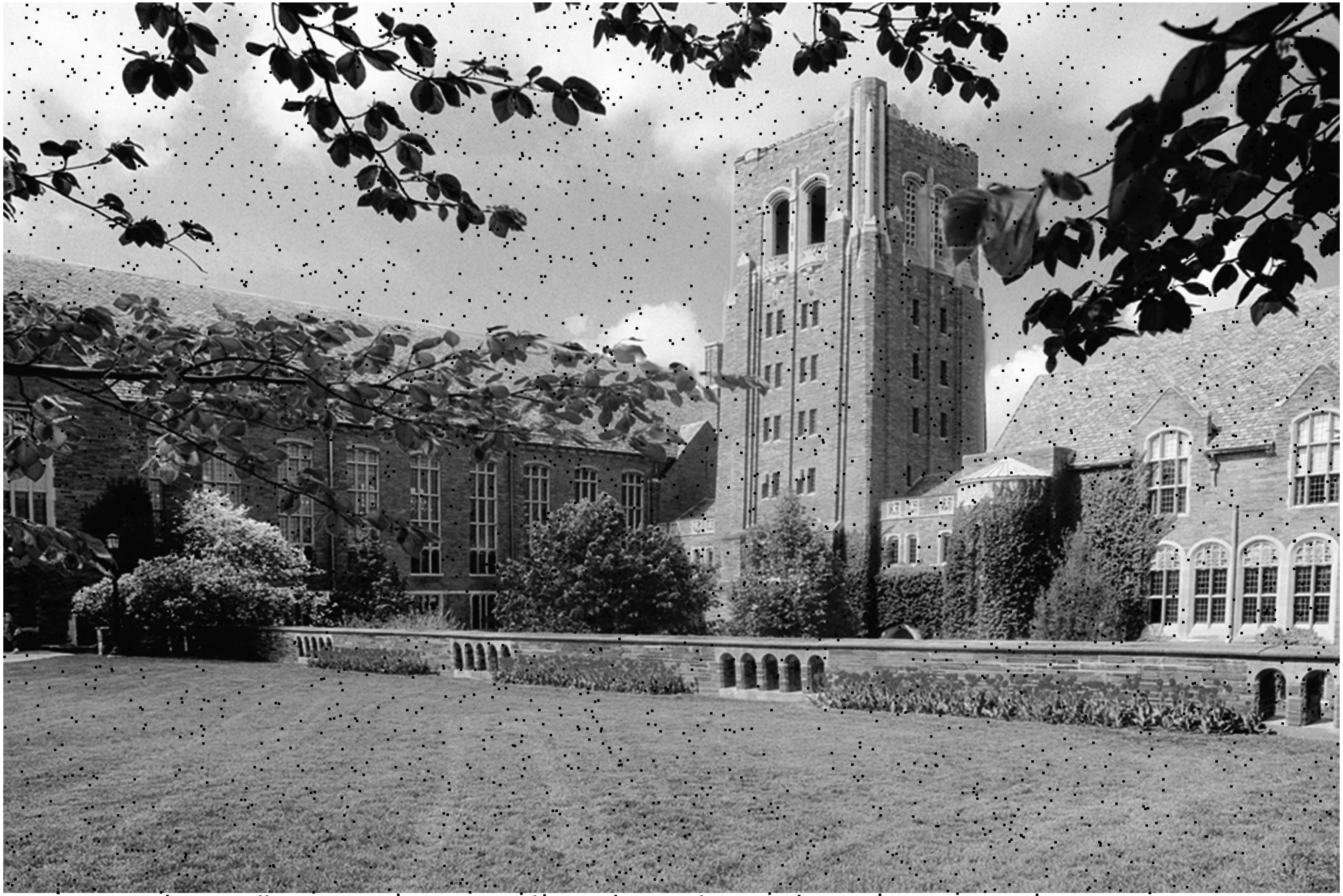


What?

The Median Filtering Problem

Visit each pixel in a picture and replace its value by the median value of its "neighbors".

A Picture With Dirt Specks



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After Median Filtering is Applied

Why?

Each Problem has a couple of Python "nuggets" to practice with.

Each problem has something to "say" about averaging.

Each problem has a high-level "message"

A Nice Way to Wrap Up

The Apportionment Problem

The Apportionment Problem

How do you distribute 435 Congressional seats among the 50 states so that the ratio of population to delegation size is roughly the same from state to state?

Quite possibly one of the greatest division problems of all time!

Notation

Number of states:	`n`
State populations:	`p[0],…,p[n-1]`
State delegation size:	`d[0],…,d[n-1]`
Total Population:	`P`
Total number of seats:	`D`

Ideal: Equal Representation

Number of states:	`n`
State populations:	`p[0],…,p[n-1]`
State delegation size:	`d[0],…,d[n-1]`
Total Population:	`P`
Number of seats:	`D`

$$\frac{P}{D} = \frac{p[0]}{d[0]} = \ldots = \frac{p[49]}{d[49]}$$

i.e.,

$$d[k] = \frac{p[k]}{P} D$$

And so for NY in 2010..

$$NY : \frac{19421055}{309239463} 435 = 27.13$$

But delegation size must be a whole number!!!

More Realistic…

Number of states:	`n`
State populations:	`p[0],…,p[n-1]`
State delegation size:	`d[0],…,d[n-1]`
Total Population:	`P`
Number of seats:	`D`

$$\frac{P}{D} \approx \frac{p[0]}{d[0]} \approx \ldots \approx \frac{p[49]}{d[49]}$$

Definition

An Apportionment Method determines delegation sizes d[0],…,d[49] that are whole numbers so that representation is approximately equal:

$$\frac{p[0]}{d[0]} \approx \ldots \approx \frac{p[49]}{d[49]}$$

How it Is Done

Think in terms of dealing cards.

You are the dealer.

You have 435 cards to deal to 50 people.

At the Start

Everybody gets one card...

```
N = 435
d = []
for k in range(50):
    d.append(1)
    N = N-1
```

Every state has at least one congressional district

Dealing out the Rest…

```
while N > 0:
    Let k be the index of that state
         which is most deserving of an
         additional district.
    # Increase that state's delegation
    d[k] += 1
    # Decrease what's left to deal
    N = N-1
```

Several reasonable definitions of "most deserving."

The Method of Small Divisors

At this point in the "card game" deal a district to the state having the largest quotient

$$\frac{p[k]}{d[k]}$$

<u>Tends to favor big states</u>

Implementation

```
def smallDivisor(p,d):
    """  returns an int j with the
    property that p[j]/d[j] is max.

    PreC:p and d are length-50 arrays
    of ints and the d-entries are pos.
    """
    m = 0
    for k in range(50):
        if p[k]/d[k] >= m
            m = p[k]/d[k]
            j = k
    return j
```

This is the old "Look for a max" problem

Dealing out the Rest…

```
while N > 0:
    k = smallDivisors(p,d)
    # Increase that state's delegation
    d[k] += 1
    # Decrease what's left to deal
    N = N-1
```

Several reasonable definitions of "most deserving."

The Method of Large Divisors

At this point in the "card game" deal a district to the state having the largest quotient

$$\frac{p(k)}{d(k)+1}$$

Tends to favor small states

Dealing out the Rest…

```
while N > 0:
     k = largeDivisors(p,d)
     # Increase that state's delegation
     d[k] += 1
     # Decrease what's left to deal
     N = N-1
```

The Method of Major Fractions

At this point in the "card game" deal a district to the state having the largest value of

$$\frac{1}{2}\left[\frac{p(k)}{d(k)}+\frac{p(k)}{d(k)+1}\right]$$

Several reasonable definitions of "most deserving."

Dealing out the Rest…

```
while N > 0:
    k = majorFractions(p,d)
    # Increase that state's delegation
    d[k] += 1
    # Decrease what's left to deal
    N = N-1
```

The Method of Equal Proportions

At this point in the "card game" deal a district to the state having the largest value of

$$\sqrt{\frac{p(k)}{d(k)} * \frac{p(k)}{d(k)+1}}$$

This method is in use today.

Compromise via the Geometric Mean

Dealing out the Rest…

```
while N > 0:
    k = equalProportions(p,d)
     # Increase that state's delegation
    d[k] += 1
    # Decrease what's left to deal
    N = N-1
```

Four Different Ways to Compute "Most Deserving"

$$\frac{p(k)}{d(k)}$$

$$\frac{p(k)}{d(k)+1}$$

$$\frac{1}{2}\left[\frac{p(k)}{d(k)}+\frac{p(k)}{d(k)+1}\right]$$

$$\sqrt{\frac{p(k)}{d(k)}*\frac{p(k)}{d(k)+1}}$$

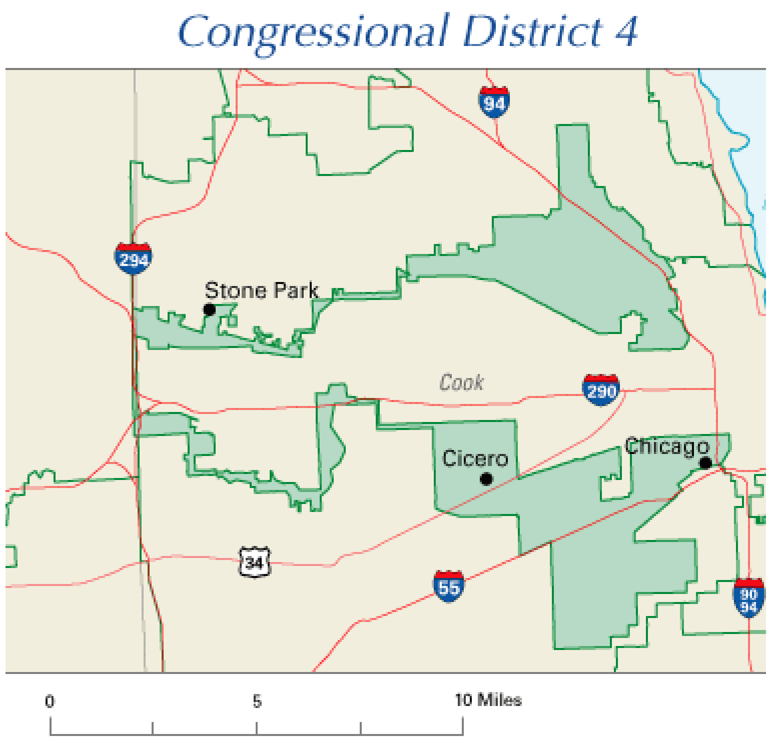
And two different ways to compute an average

Takeaway: There is a Subjective Component to Math+Computing

One can design more equitable methods for apportionment, but they are complicated and cannot be "sold" to the lay public.

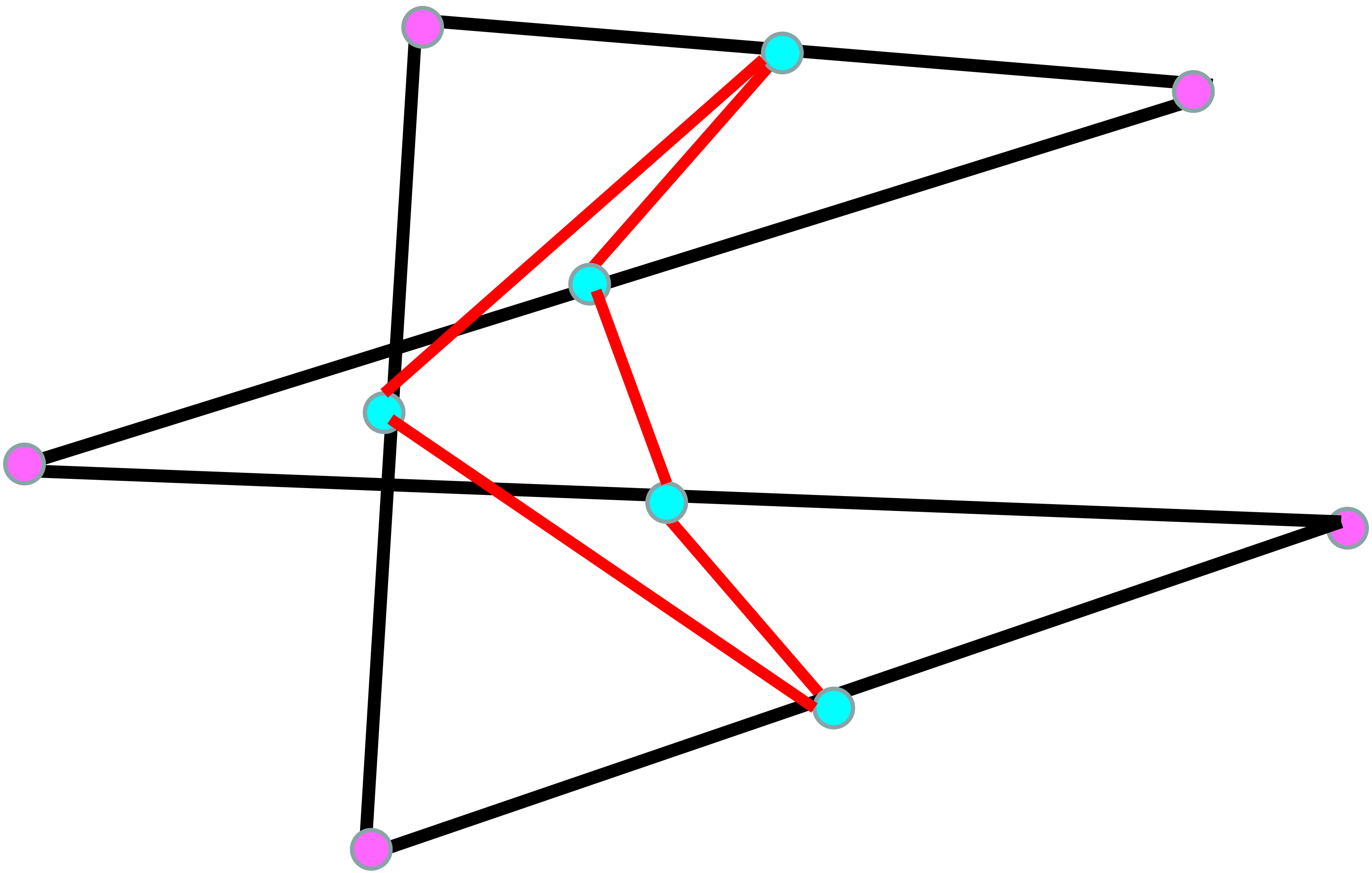
Another Division Problem

Gerrymandering: The Art of drawing district boundaries so as to favor incumbents



Polygon Averaging

Connect the Midpoints



A Useful Class

```
class polygon:
    def __init__(self,x,y):
        self.x = x
        self.y = y
```

x and y are numpy arrays that name the vertices of the polygon:

```
(x[0],y[0]),…,(x[n-1],y[n-1])
```

The New Polygon

```
def newPoly(self):
    n = len(self.x); x = zeros(n);y = zeros(n)
    for k in range(n):
        # Get the next midpoint.
        j = (k+1)%n
        x[k] = (self.x[k]+self.x[j])/2
        y[k] = (self.y[k]+self.y[j])/2
    self.x = x
    self.y = y
```

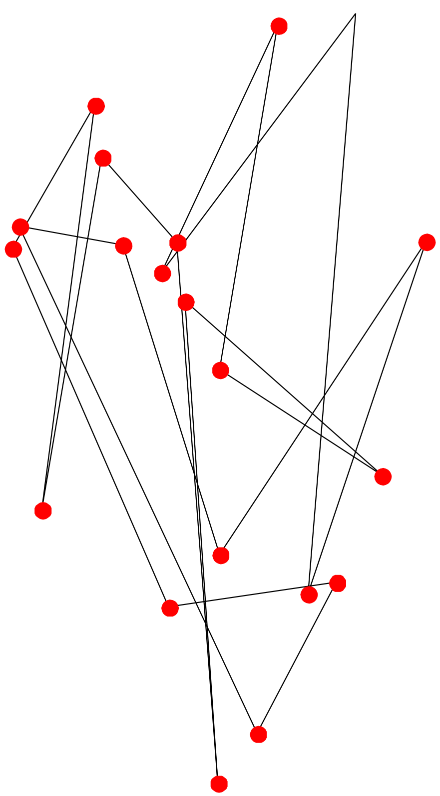
Order From Chaos

1. Pick n, say n= 30
2. Generate random lists of floats x and y
3. P = polygon(x,y)
4. Then repeatedly replace P by with a new polygon obtained by connecting midpoints:

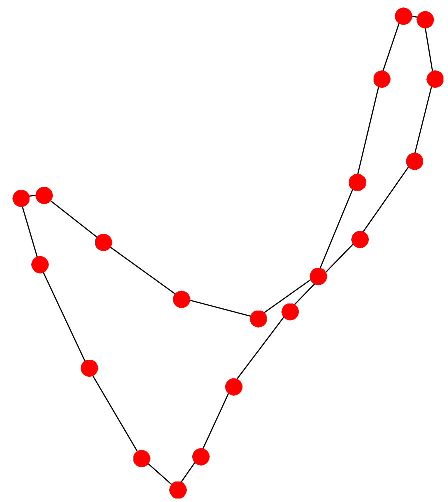
```
for i in range(200):
    P.newPolygon()
    P.plotPoly()
```

The Polygon Untangles Itself and Heads Towards an Ellipse

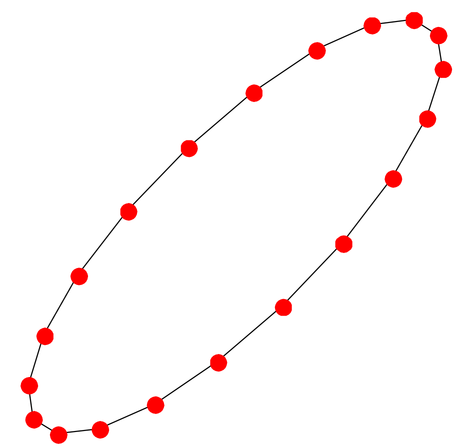
At the Start



After 40 iterations

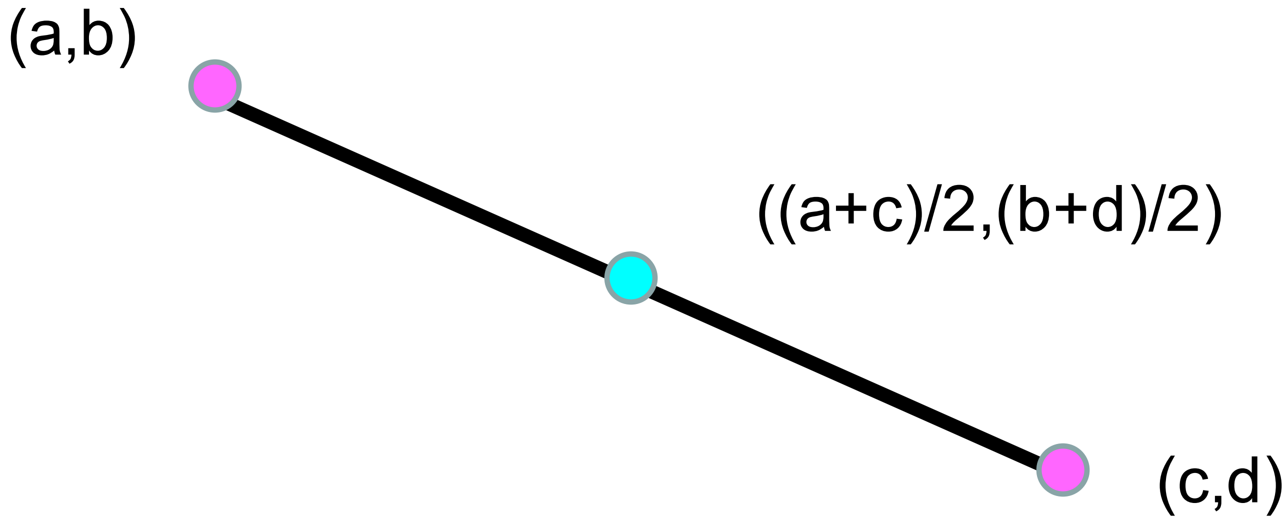


After 200 iterations



It's About Repeated Averaging

A midpoint is the average of the endpoints.



Median Filtering

Pictures as Arrays

A black and white picture can be encoded as a 2-dimensional array of numbers

Typical:

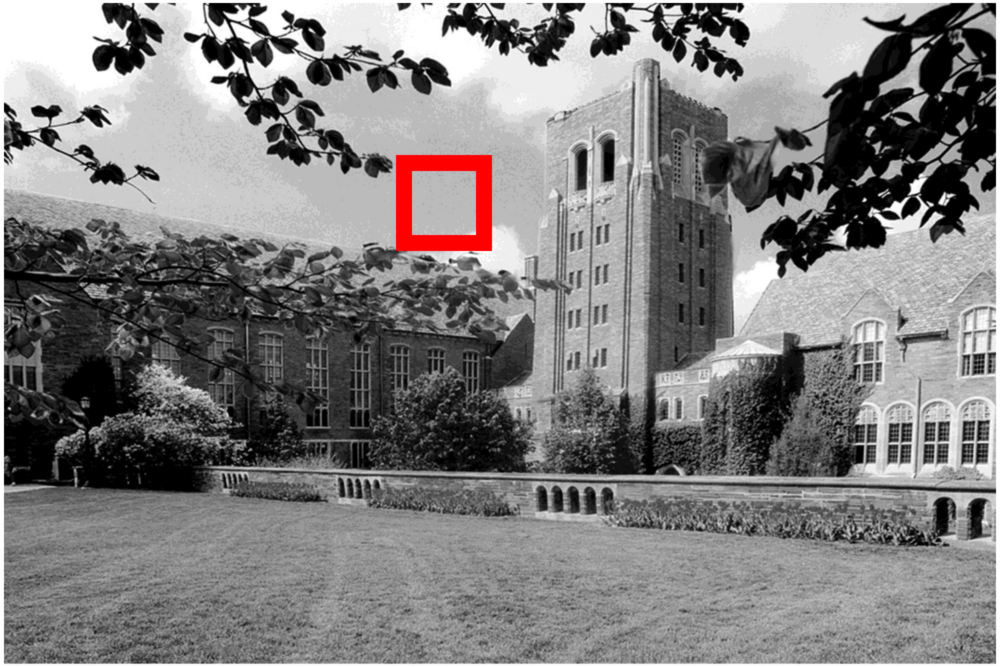
```
0    <=    A[i,j]    <=   255
```

(black) (white)

Values in between correspond to different levels of grayness.

Just a Bunch of Numbers

1458-by-2084



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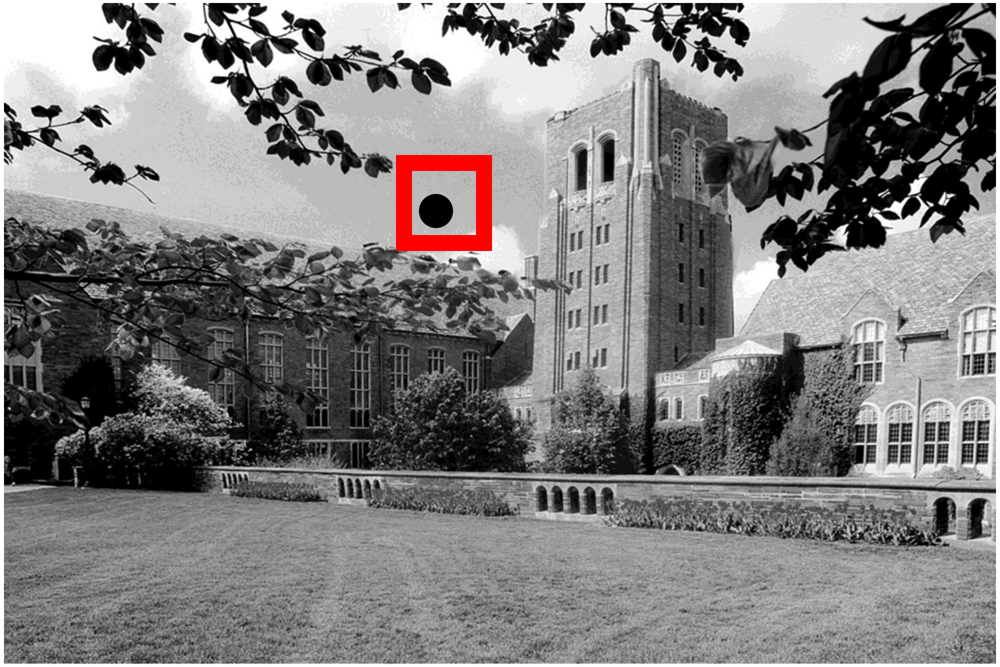
150	149	152	153	152	155
151	150	153	154	153	156
153	151	155	156	155	158
154	153	156	157	156	159
156	154	158	159	158	161
157	156	159	160	159	162

Dirt!

1458-by-2084

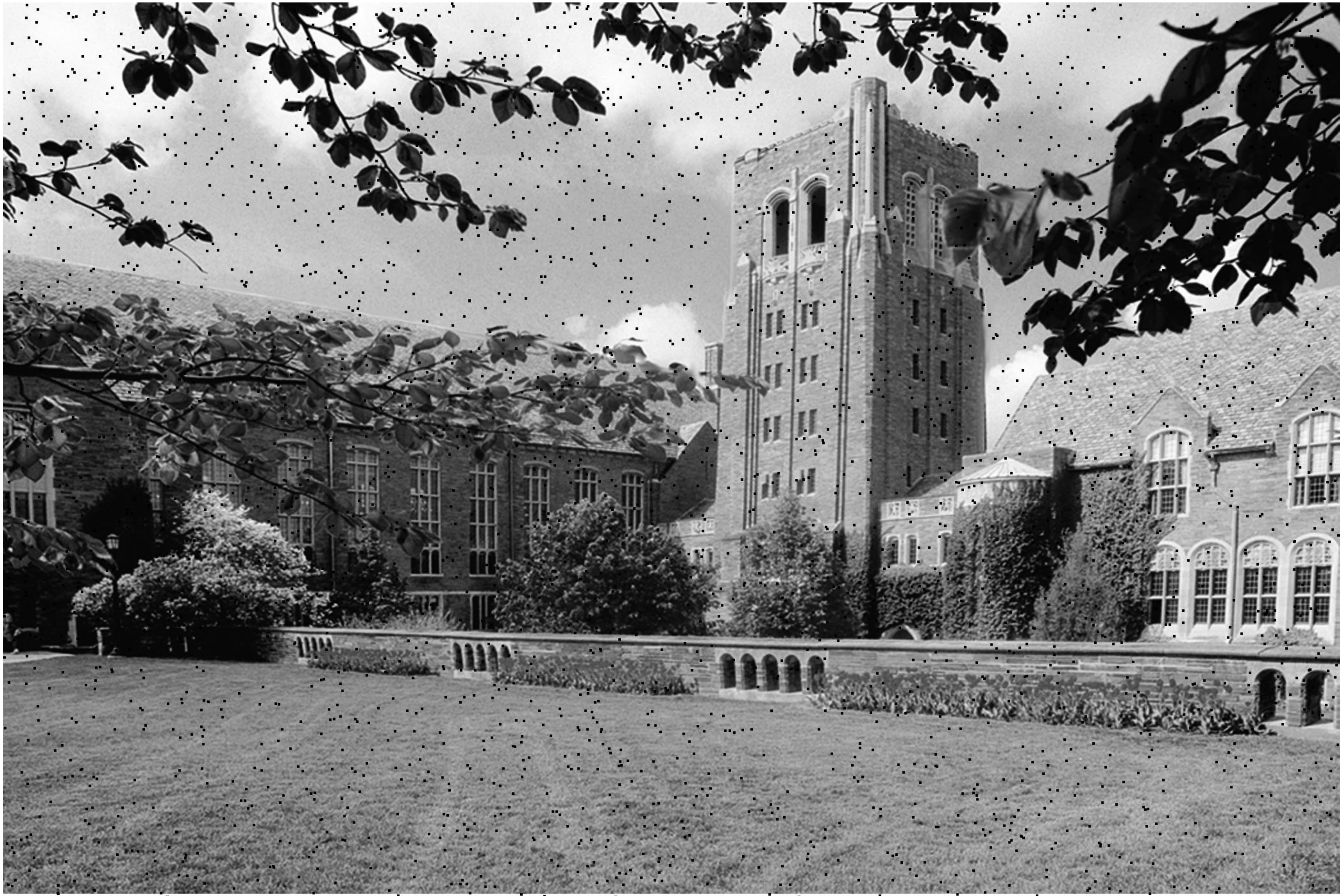


Cornell University Law School
Photograph by Cornell University Photography

150	149	152	153	152	155
151	150	153	154	153	156
153	2	3	156	155	158
154	2	1	157	156	159
156	154	158	159	158	161
157	156	159	160	159	162

Note how the "dirty pixels" look out of place

Can We Filter Out the "Noise"?



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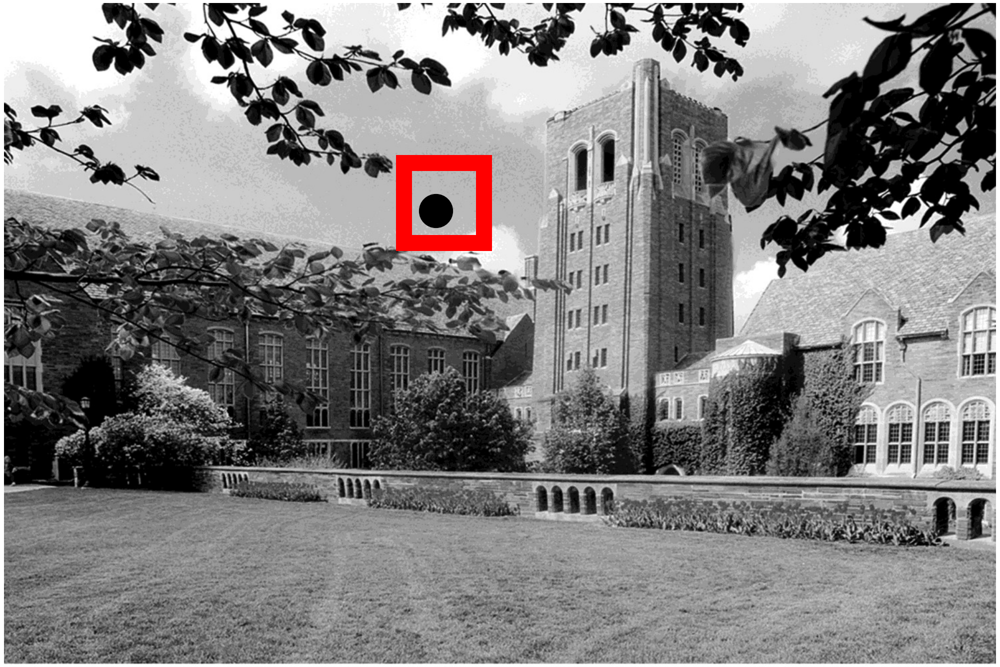
Idea

1458-by-2084



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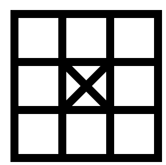
150	149	152	153	152	155
151	150	153	154	153	156
153	?	?	156	155	158
154	?	?	157	156	159
156	154	158	159	158	161
157	156	159	160	159	162

Assign "typical" neighborhood gray values to "dirty pixels"

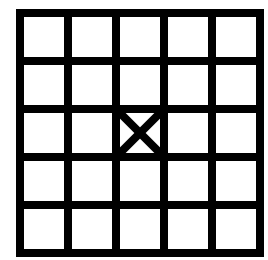
Getting Precise

"Typical neighborhood gray values"

Could use Median Or Mean



radius 1



radius 3

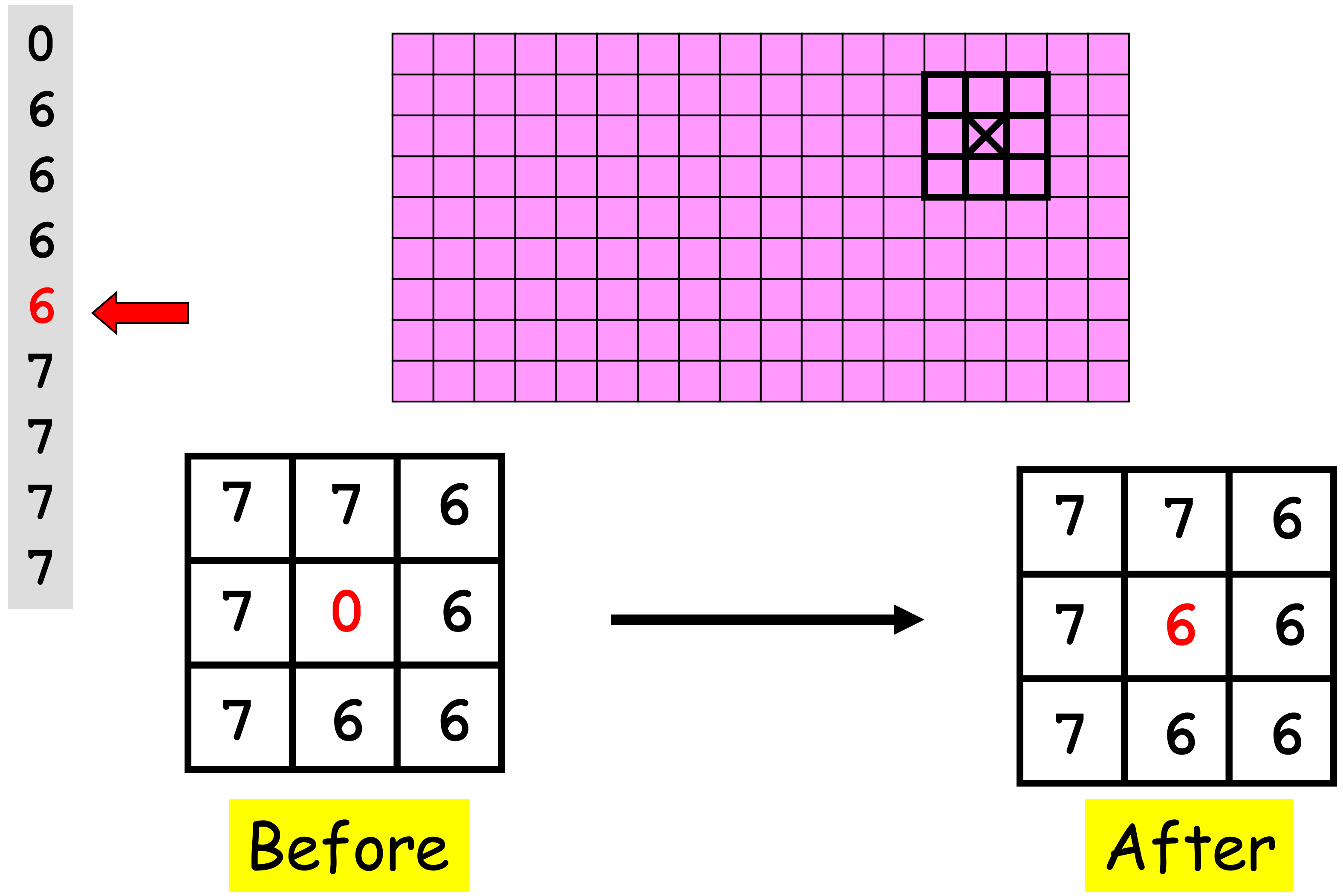
We'll look at "Median Filtering" first…

Median Filtering

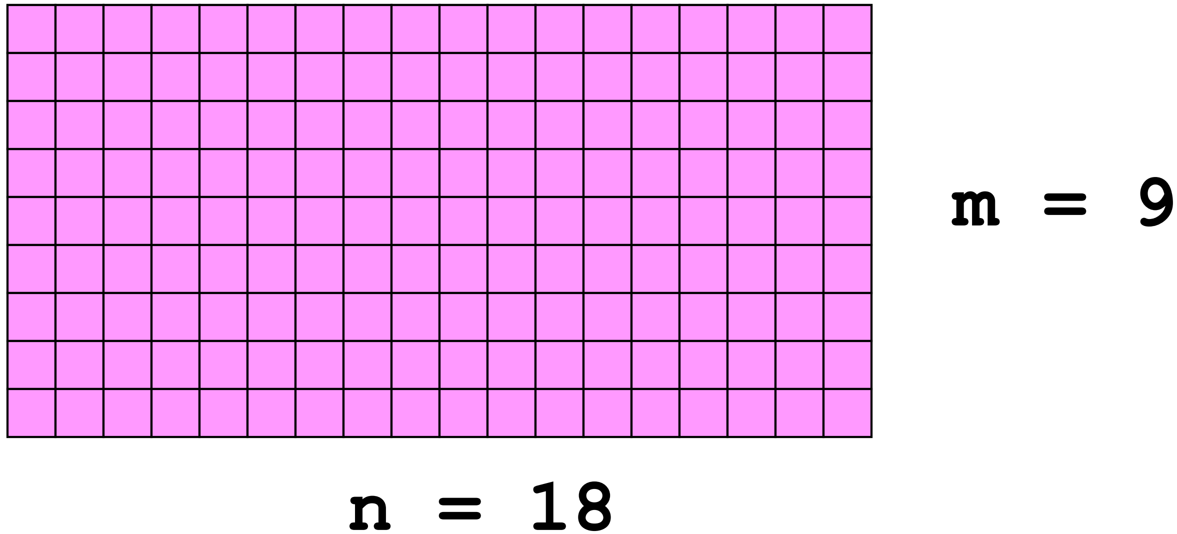
Visit each pixel.

Replace its gray value by the median
of the gray values in the "neighborhood".

Using a radius 1 “Neighborhood”



How to Visit Every Pixel



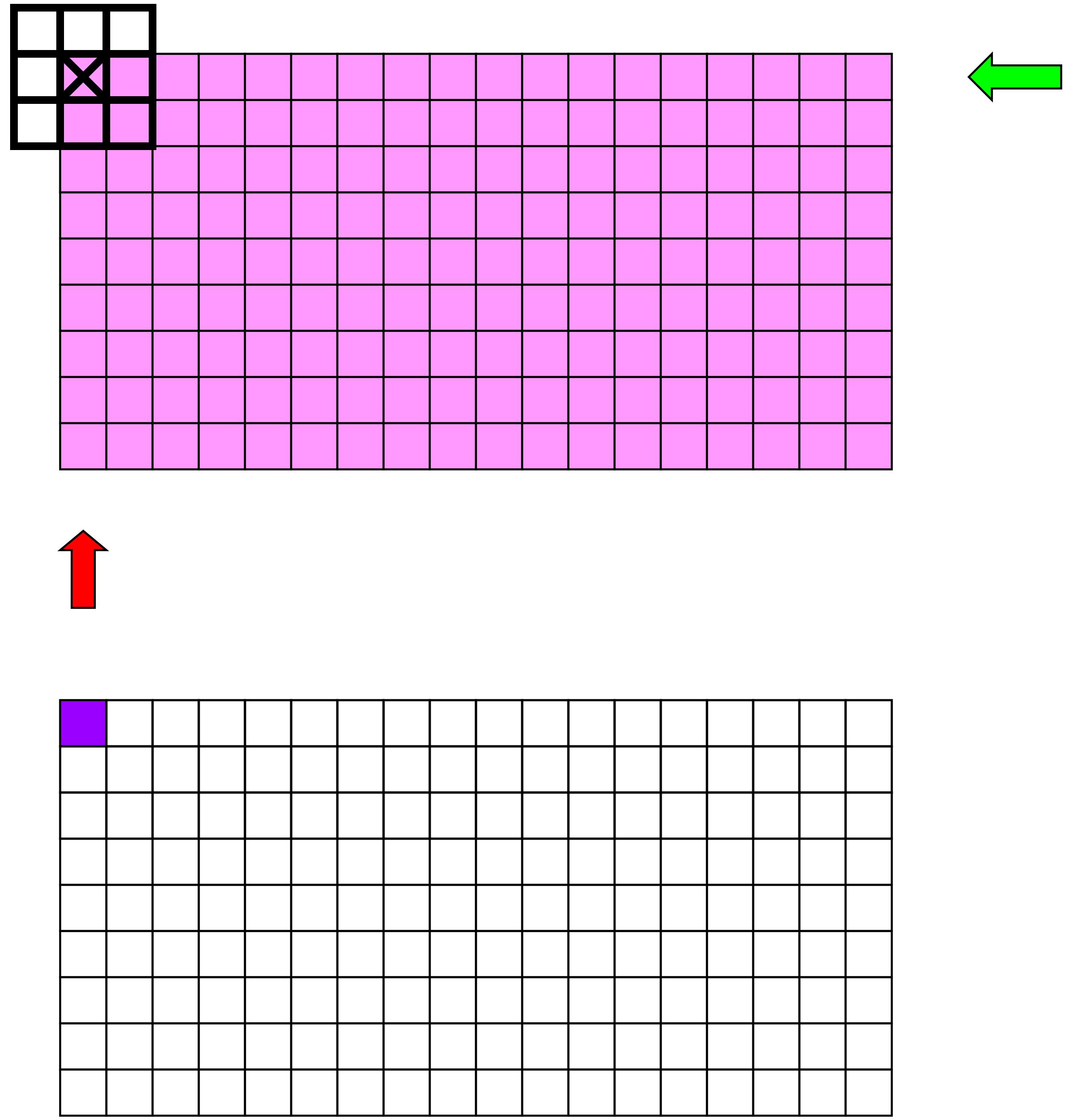

```
for i in range(m):
    for j in range(n):
        Compute new gray value for pixel (i,j).
```

Original:

i = 0

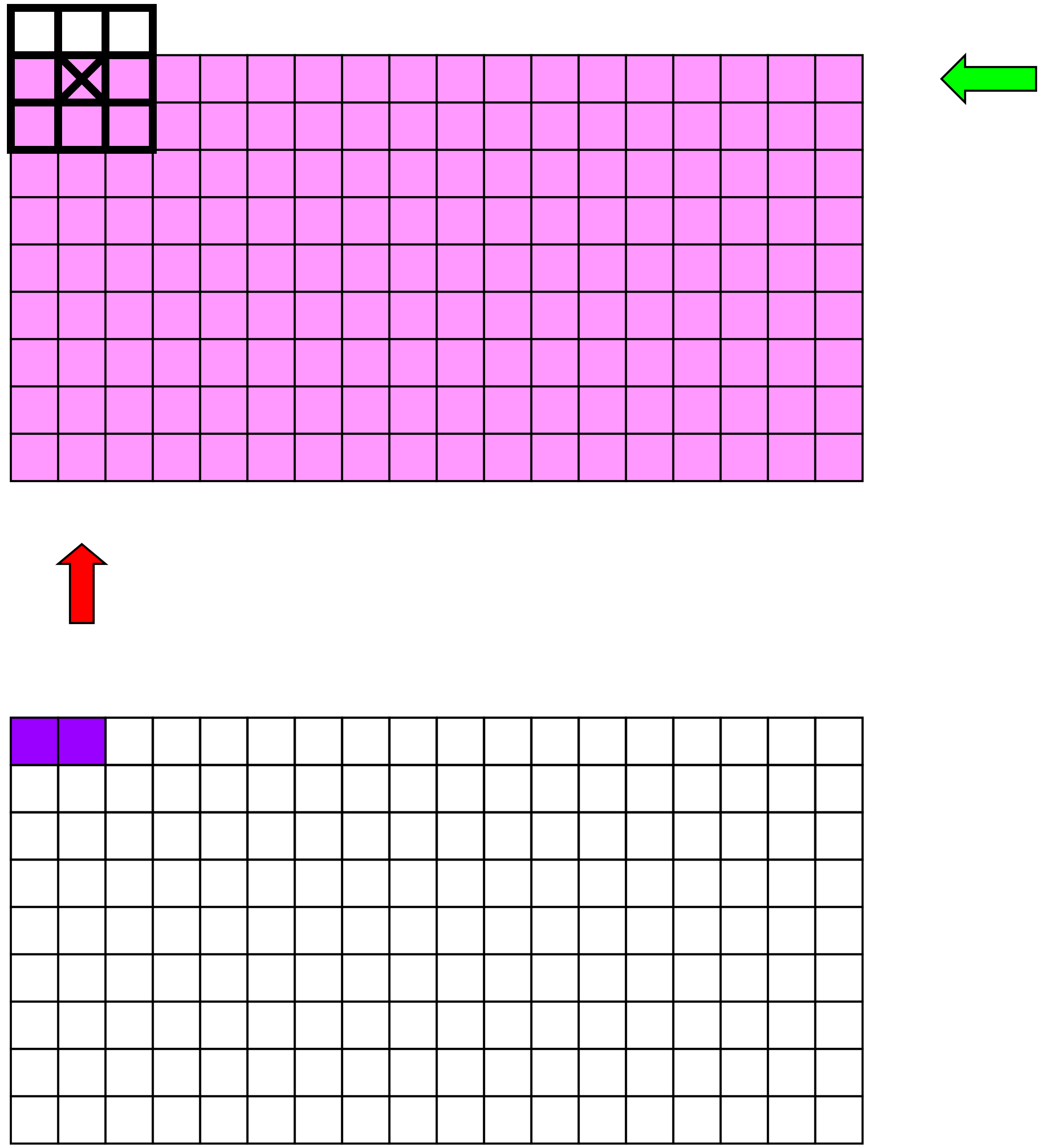
j = 0

Filtered:



Replace ⊠ with the median of the values under the window.

Original:



```
i = 0
```

```
j = 1
```

Filtered:

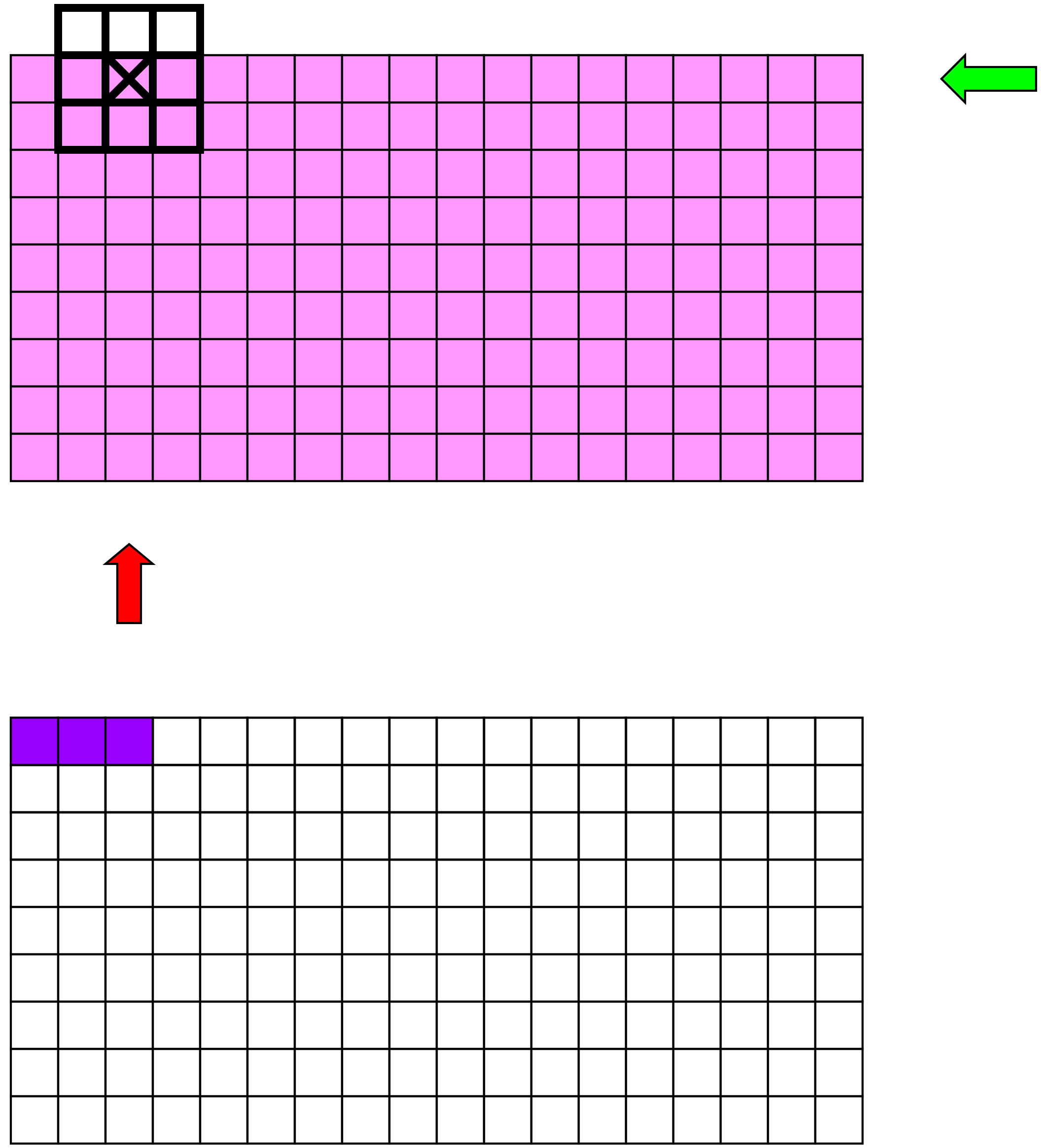
Replace ☒ with the median of the values under the window.

Original:

i = 0

j = 2

Filtered:



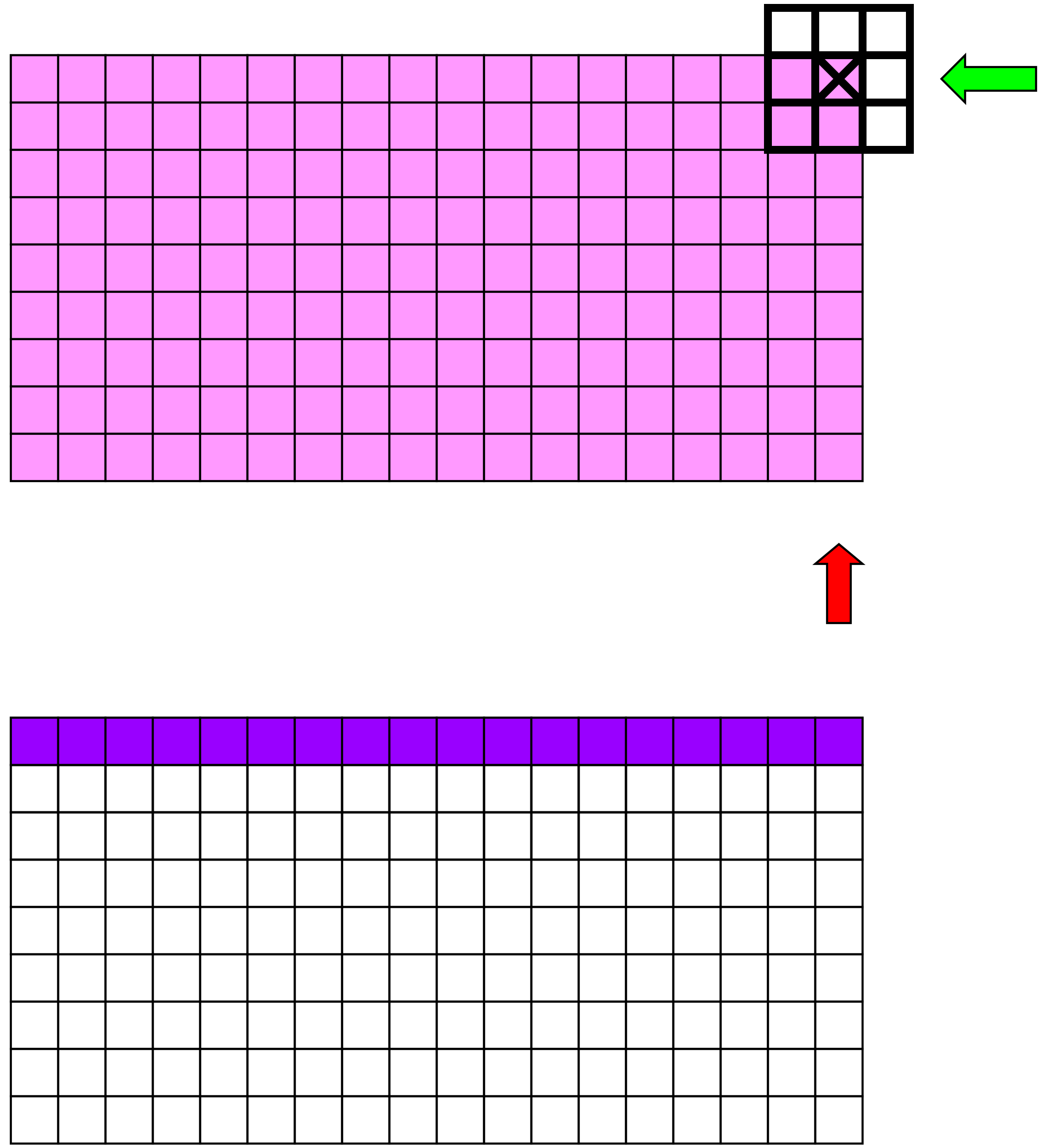
Replace ⊠ with the median of the values under the window.

Original:

`i = 0`

`j = n-1`

Filtered:



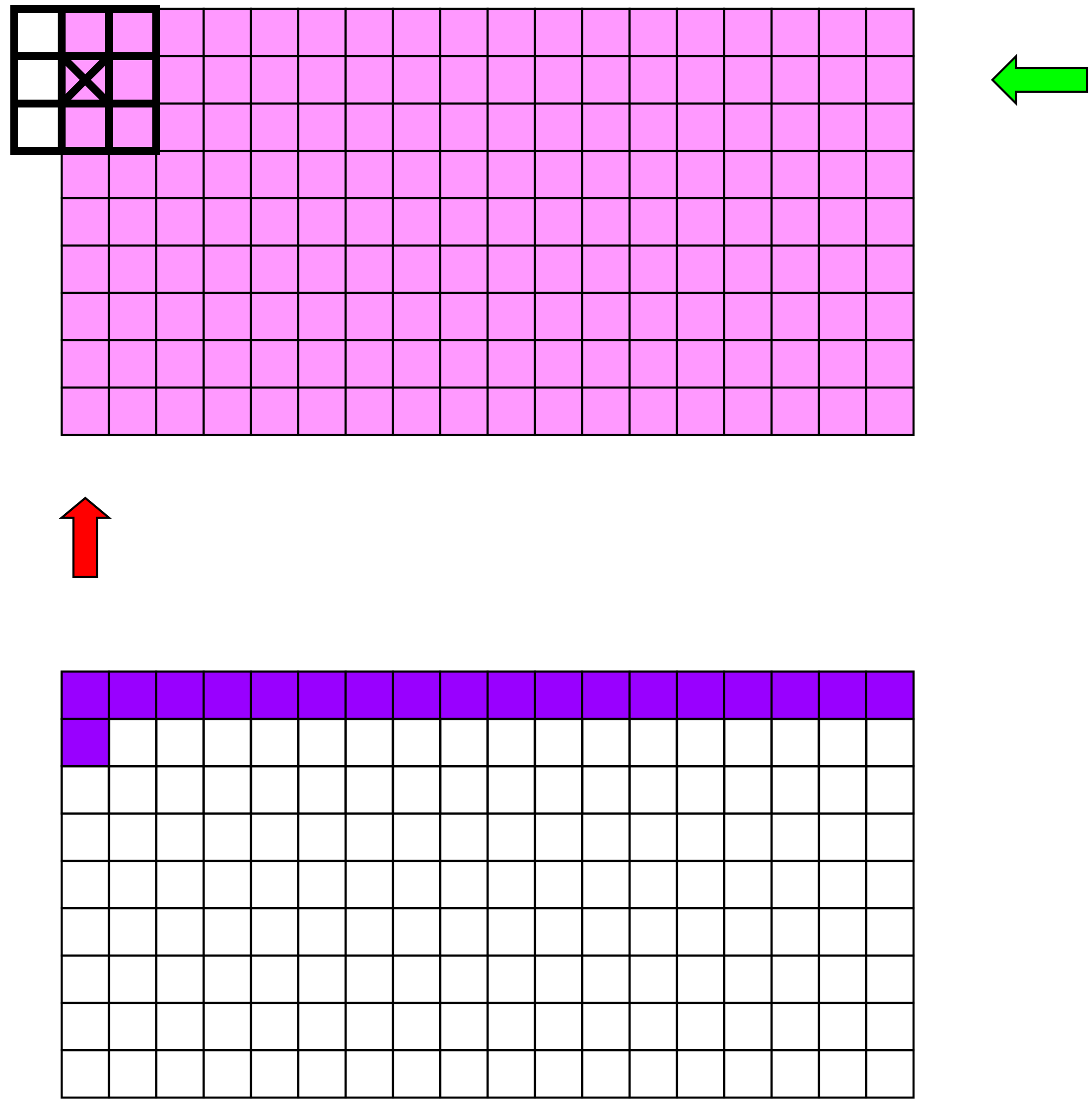
Replace ⊠ with the median of the values under the window.

Original:

`i = 1`

`j = 0`

Filtered:



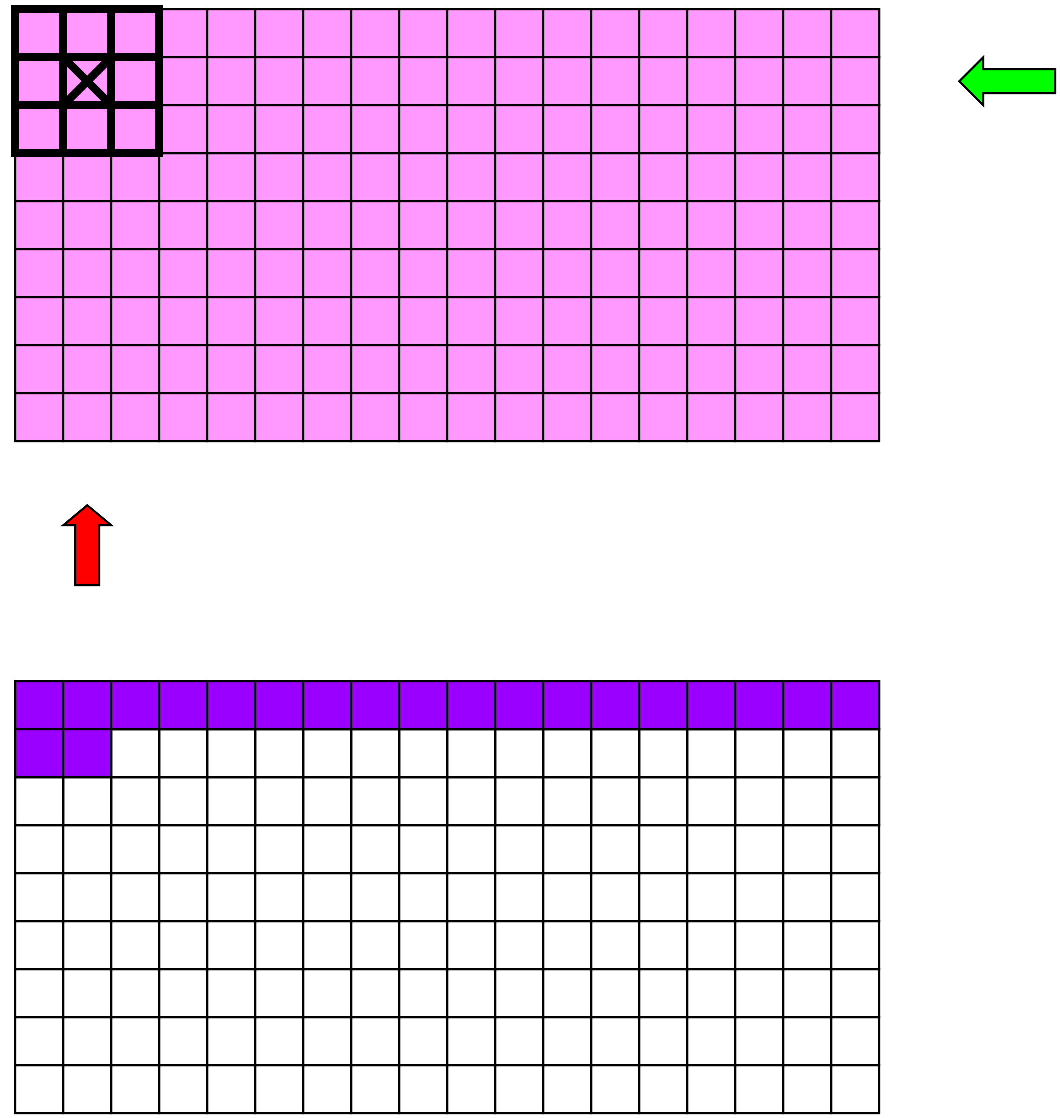
Replace ⊠ with the median of the values under the window.

Original:

i = 1

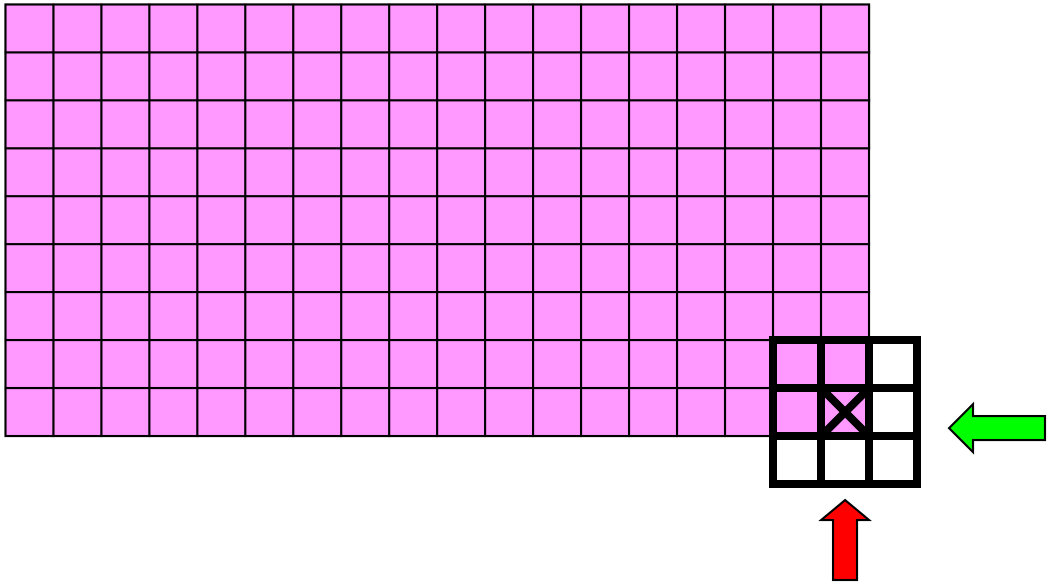
j = 1

Filtered:



Replace ☒ with the median of the values under the window.

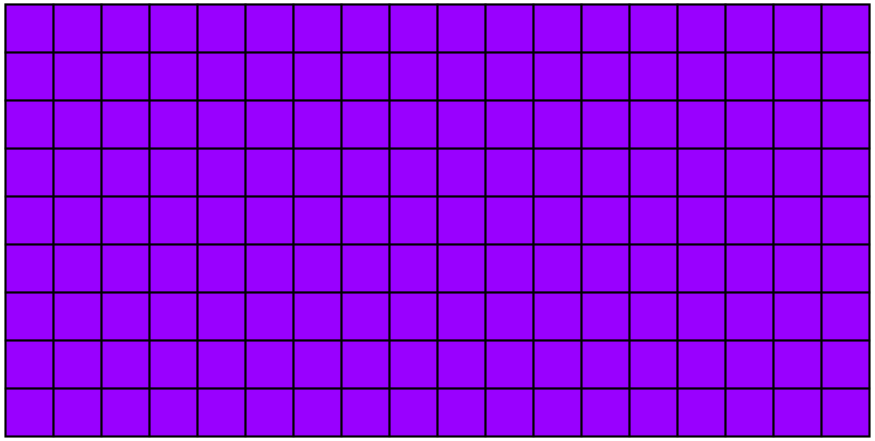
Original:



```
i = m-1
```

```
j = n-1
```

Filtered:



Replace ☒ with the median of the values under the window.

What We Need…

(1) A function that computes the median value in a 2-dimensional array *C*:

```
m = medVal(C)
```

(2) A function that builds the filtered image by using median values of radius r neighborhoods:

```
B = medFilter(A,r)
```

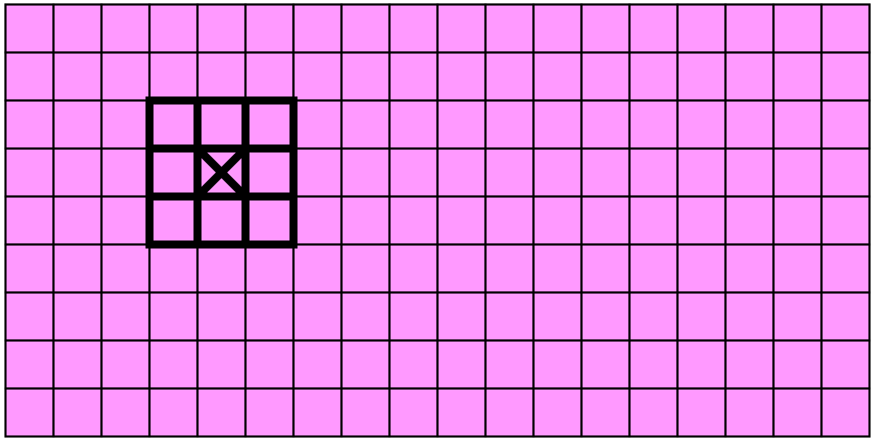
Medians vs Means

```
A =
   150   151   158   159   156
   153   151   156   155   151
   150   155   152   154   159
   156   154   152   158   152
   152   158   157   150   157

Median = 154   Mean = 154.2
```

Medians vs Means

```
A =
   150   151   158   159   156
   153   151   156   155   151
   150   155     0   154   159
   156   154   152   158   152
   152   158   157   150   157

Median = 154  Mean = 148.2
```

Back to Filtering...



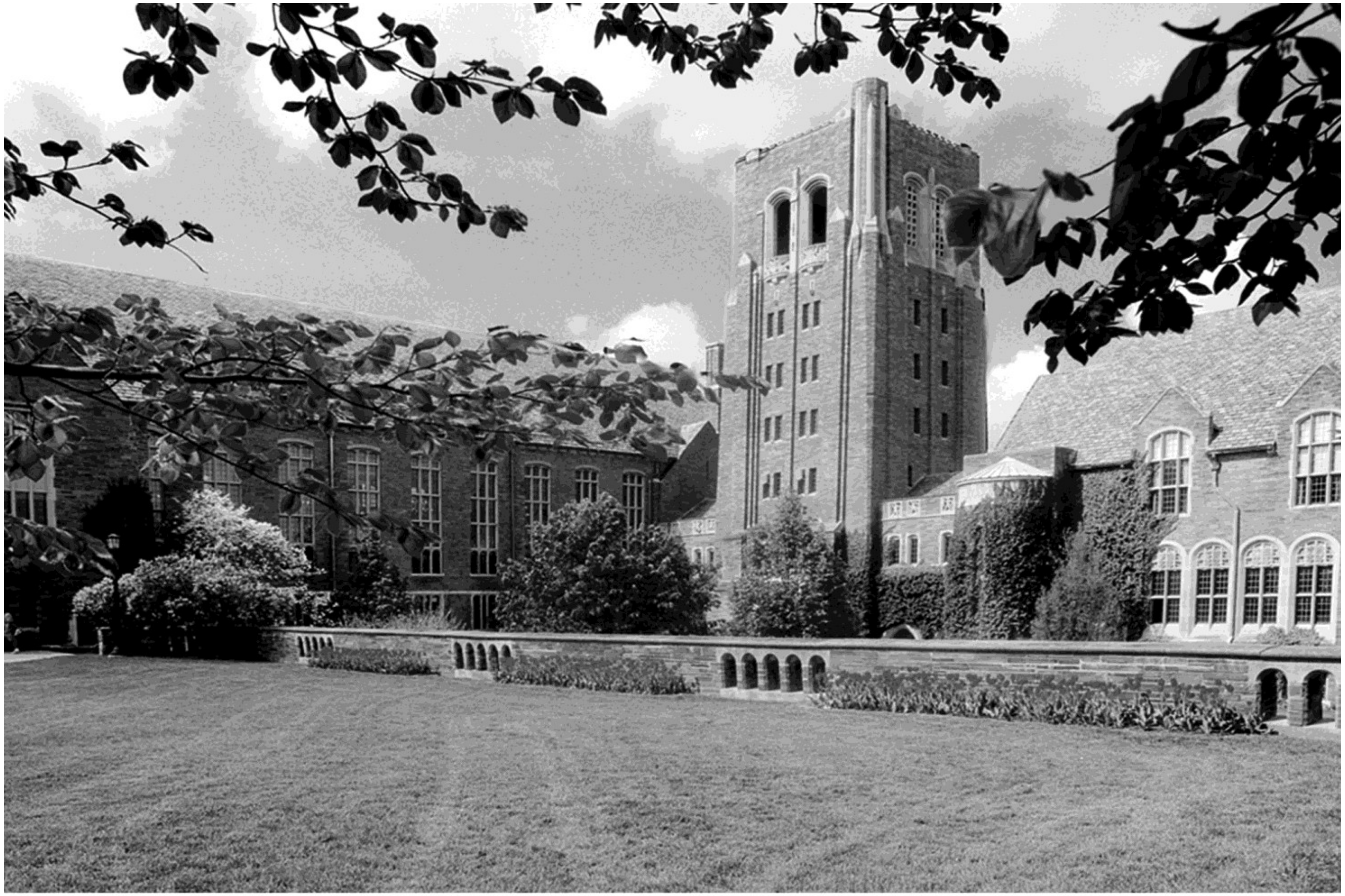
m = 9

n = 18

```
for i in range(m):
    for j inrange(n)
        Compute new gray value for pixel (i,j).
    end
end
```

B = medFilter(A)

Original



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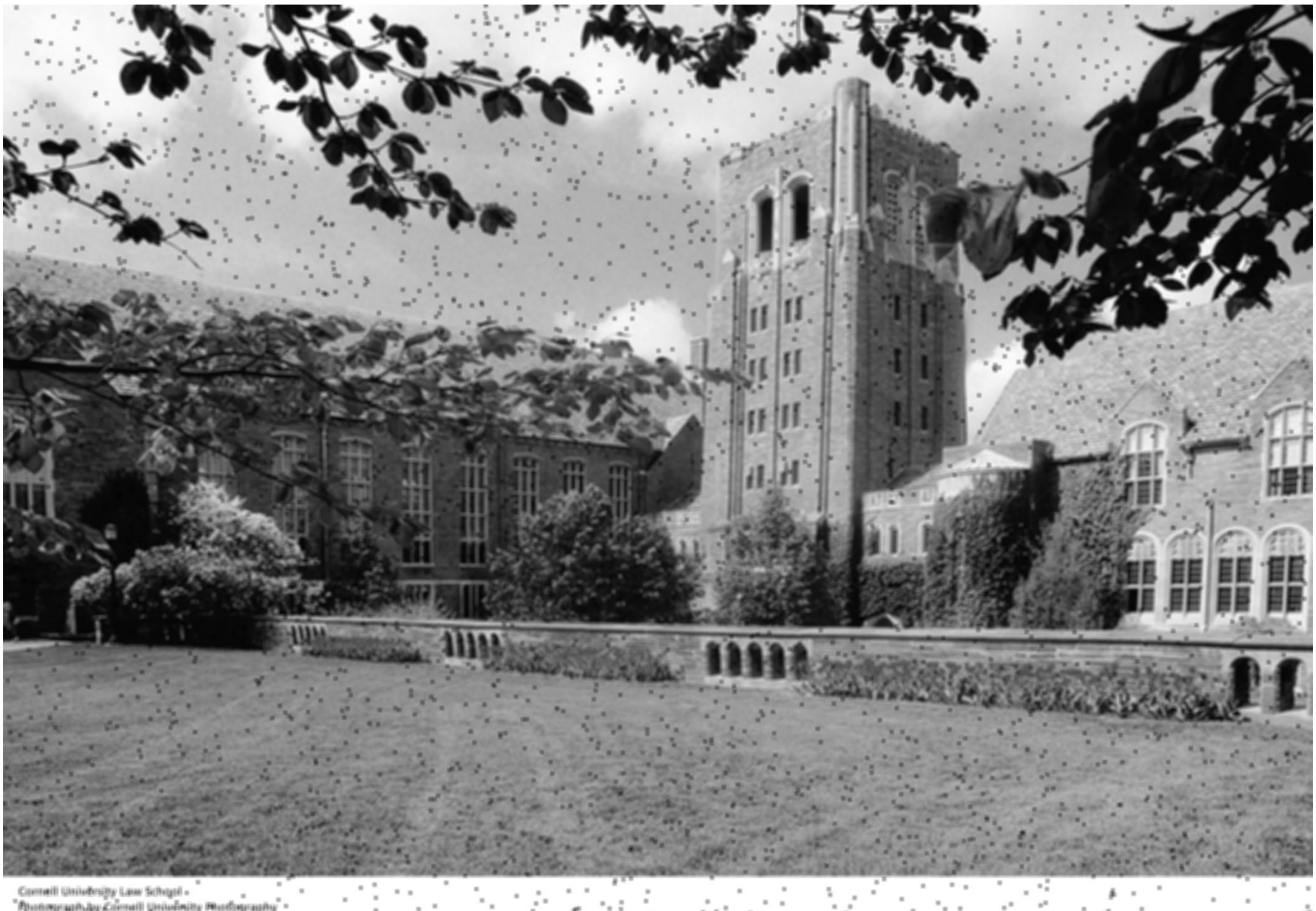
What About Using the Mean instead of the Median?

Replace each gray value with the average gray value in the radius r neighborhood.

Mean Filter with r = 3



Mean Filter with r = 10

Why it Fails

150	149	152	153	152	155
151	150	153	154	153	156
153	2	3	156	155	158
154	2	1	157	156	159
156	154	158	159	158	161
157	156	159	160	159	162

85	86
87	88

The mean does not capture representative values.

And Median Filters Leave Edges (Pretty Much) Alone

200	200	200	200	200	200
200	200	200	200	200	100
200	200	200	200	100	100
200	200	200	100	100	100
200	200	100	100	100	100
200	100	100	100	100	100

Inside the box, the 200's stay at 200 and the 100's stay at 100.

Takeaways

Image processing is all about operations on 2-dimensional arrays.

Simple operations on small patches are typically repeated again and again

There is a profound difference between the median and the mean when filtering noise