Lecture 15

More Recursion

Announcements for This Lecture

Prelim 1

- Prelim 1 available
  - Pick up in Lab Section
  - Solution posted in CMS
  - **Mean**: 75.8, **Median**: 79
- What are letter grades?
  - Way too early to tell
  - **A**: Could be a consultant
  - **B**: Could take 2110
  - **C**: Good enough to pass

Assignments and Labs

- Need to be working on A4
  - Instructions are posted
  - Just reading it takes a while
  - Slightly longer than A3
  - Problems are harder
- **Lab Today**: lots of practice!
  - 4 functions are mandatory
  - Lots of optional ones to do
  - Exam questions on Prelim 2

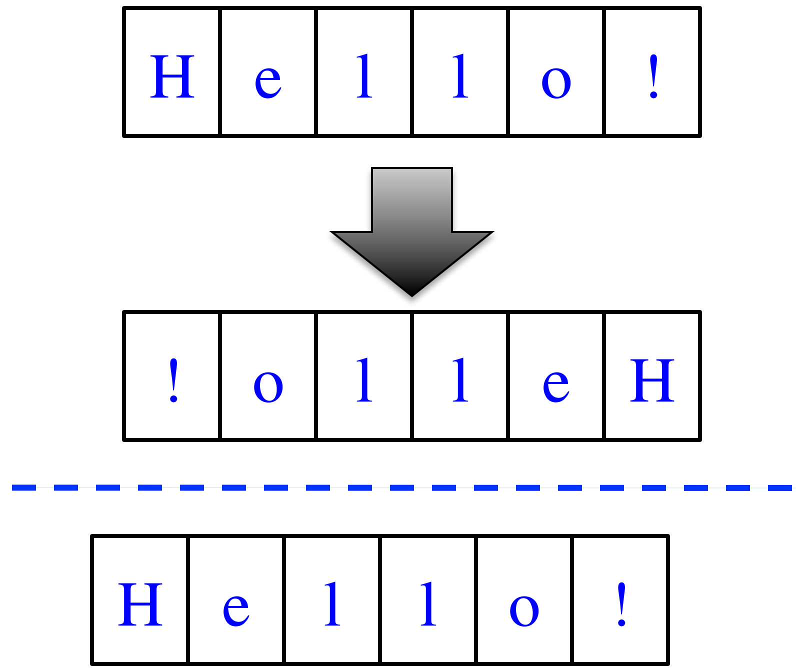
Recursion

- **Recursive Definition**:
  A definition that is defined in terms of itself
- **Recursive Function**:
  A function that calls itself (directly or indirectly)

- Powerful programming tool
  - Want to solve a difficult problem
  - Solve a simpler problem instead
- **Goal of Recursion:**
  Solve original problem with help of simpler solution

Example: Reversing a String

- **Precise Specification:**
  - Returns: reverse of s
- Solving with recursion
  - Suppose we can reverse a smaller string (e.g. less one character)
  - Can we use that solution to reverse whole string?
- Often easy to understand first without Python
  - Then sit down and code



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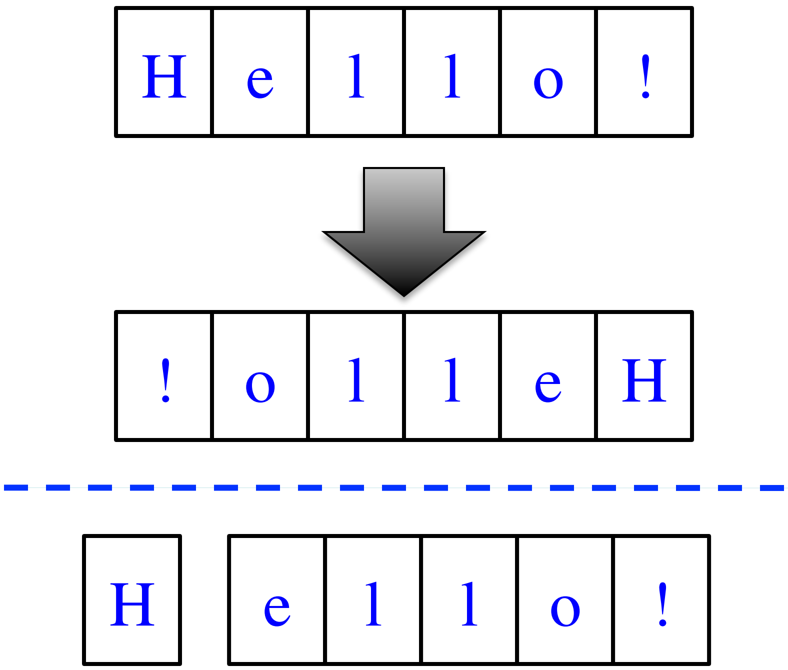

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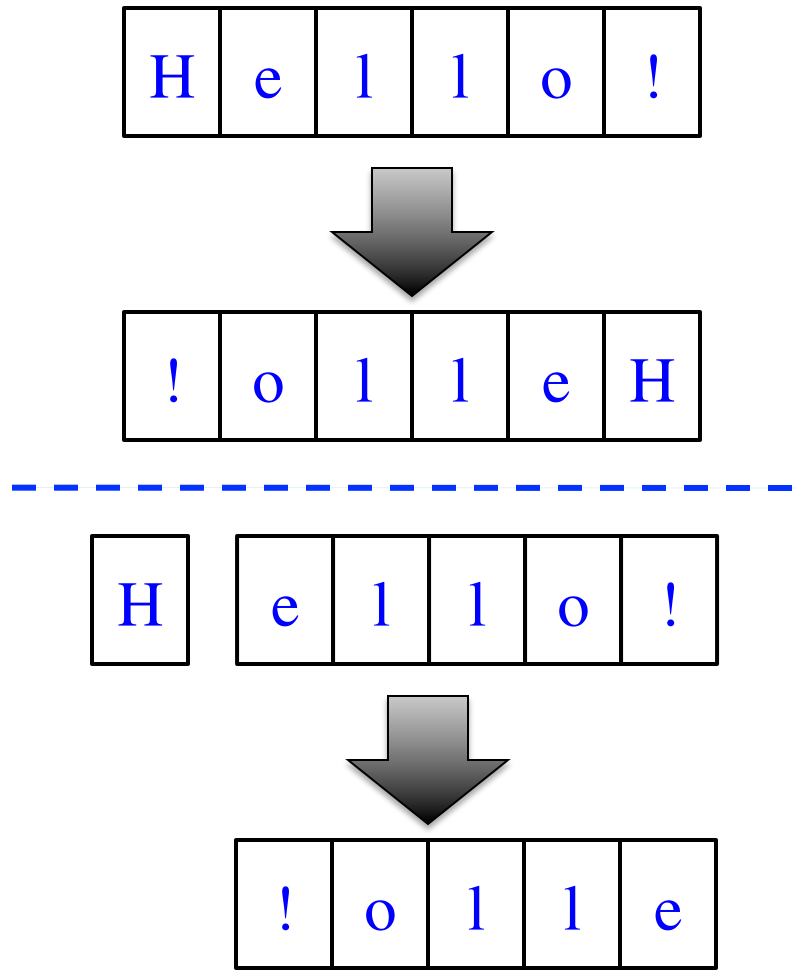

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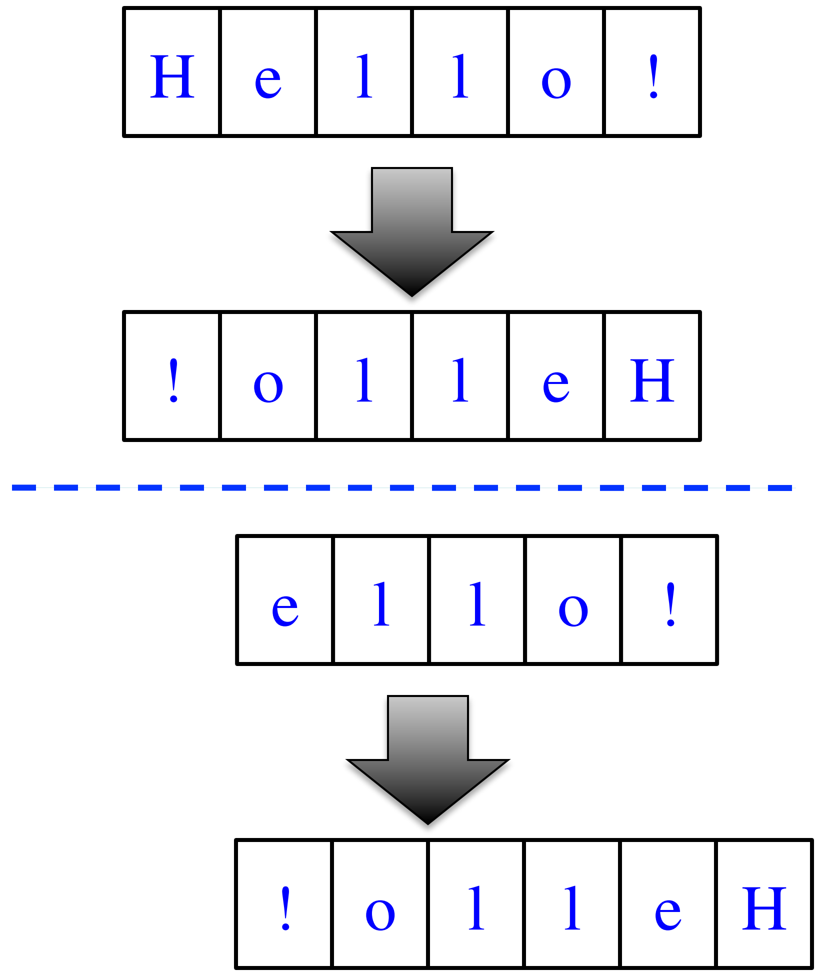
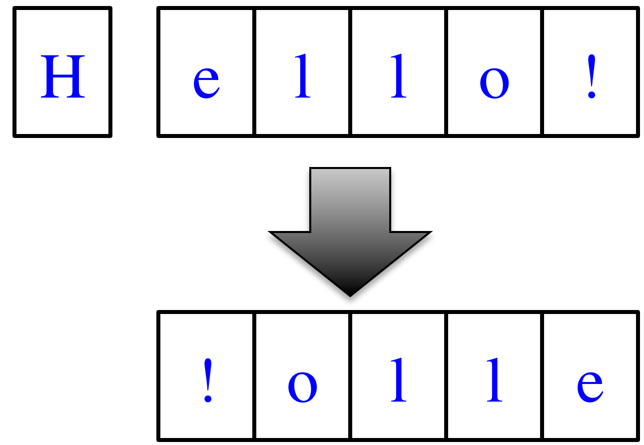


Example: Reversing a String

```
def reverse(s):
    """Returns: reverse of s

    Precondition: s a string"""
    # {s is empty}
    if s == '':
        return s

    # { s at least one char }
    # (reverse of s[1:])+s[0]
    return reverse(s[1:])+s[0]
```



✔ 1. Precise specification?
✔ 2. Base case: correct?
✔ 3. Recursive case: progress to termination?
✔ 4. Recursive case: correct?

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form a palindrome
- **Example:**

have to be the same

AMANAPLANACANALPANAMA

has to be a palindrome

- **Precise Specification:**

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
```

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are equal, and
  - the rest of the characters form a palindrome
- **Recursive Function:**

Recursive Definition

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
    if len(s) < 2:
        return True

    // { s has at least two characters }
    return s[0] == s[-1] and ispalindrome(s[1:-1])
```

Base case

Recursive case

Example: Palindromes

- String with ≥ 2 characters is a palindrome if:
  - its first and last characters are e
  - the rest of the characters form a
- **Recursive Function:**

```
def ispalindrome(s):
    """Returns: True if s is a palindrome"""
    if len(s) < 2:
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    // { s has at least two characters }
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```

Base case

Recursive case

1. Precise specification?
2. Base case: correct?
3. Recursive case: progress to termination?
4. Recursive case: correct?

Example: More Palindromes

```
def ispalindrome2(s):
    """Returns: True if s is a palindrome
    Case of characters is ignored."""
    if len(s) < 2:
        return True

    // { s has at least two characters }
    return ( equals_ignore_case(s[0],s[−1])
                 and ispalindrome2(s[1:-1]) )
```

Example: More Palindromes

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def ispalindrome2(s):
    """Returns: True if s is a palindrome
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Precise Specification

Example: More Palindromes

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    // { s has at least two characters }
    return ( equals_ignore_case(s[0],s[−1])
             and ispalindrome2(s[1:-1]) )

def equals_ignore_case (a, b):
    """Returns: True if a and b are same ignoring case"""
    return a.upper() == b.upper()
```

Precise Specification

Example: More Palindromes

```
def ispalindrome3(s):
    """Returns: True if s is a palindrome
    Case of characters and non-letters ignored."""
    return ispalindrome2(depunct(s))


def depunct(s):
    """Returns: s with non-letters removed"""
    if s == '':
        return s
    # use string.letters to isolate letters
    return (s[0]+depunct(s[1:]) if s[0] in string.letters
              else depunct(s[1:]))
```

Use helper functions!

- Often easy to break a problem into two
- Can use recursion more than once to solve

Recursion is form of Divide and Conquer

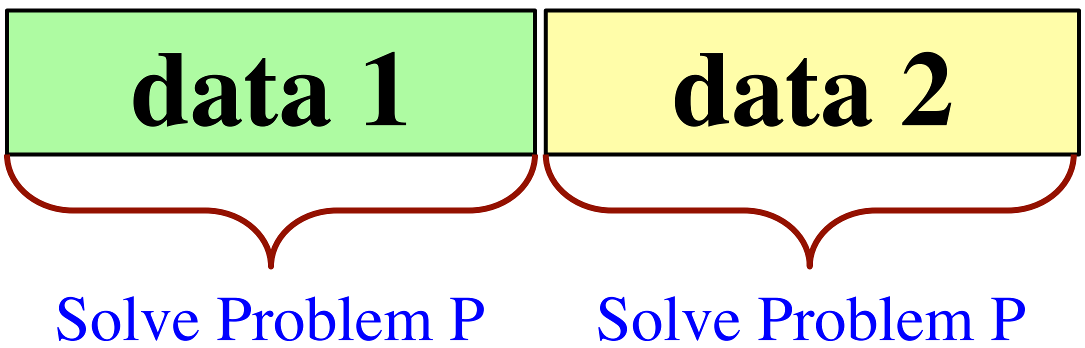
Goal: Solve problem P on a piece of data

data

Recursion is form of Divide and Conquer

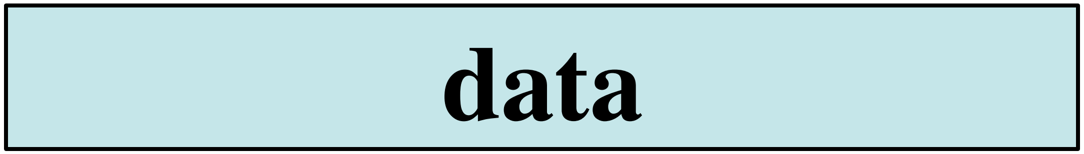
Goal: Solve problem P on a piece of data

Idea: Split data into two parts and solve problem



Recursion is form of Divide and Conquer

Goal: Solve problem P on a piece of data



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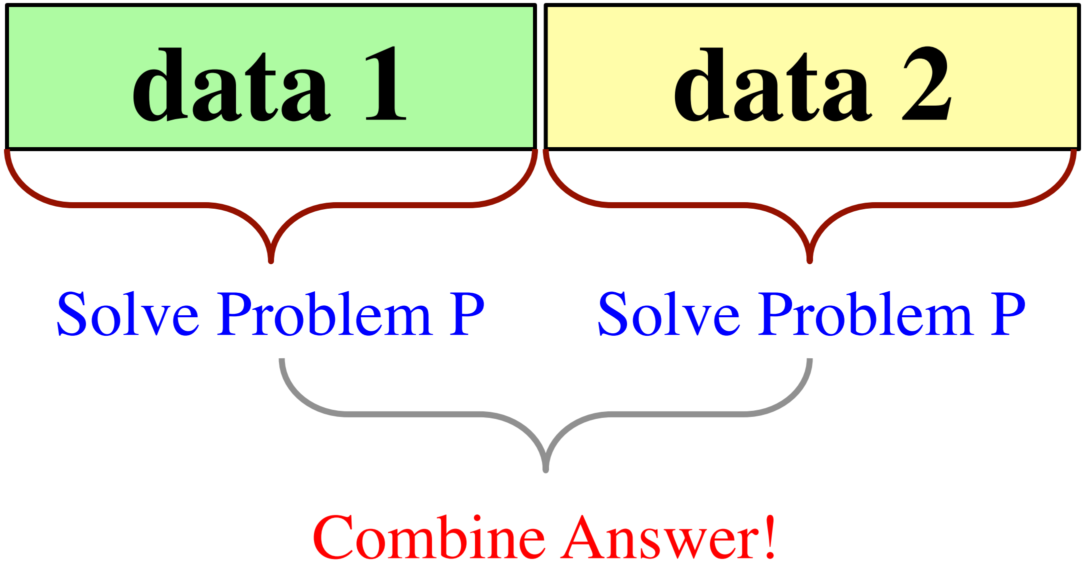

How to Break Up a Recursive Function?

```
def commafy(s):
    """Returns: string with commas every 3 digits
    e.g. commafy('5341267') = '5,341,267'
    Precondition: s represents a non-negative int"""
```

Approach 1

5	341267

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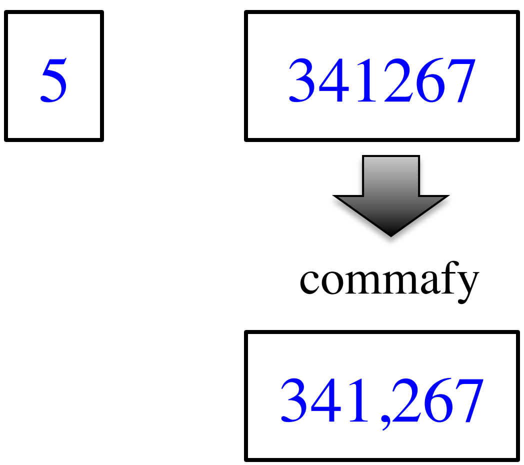

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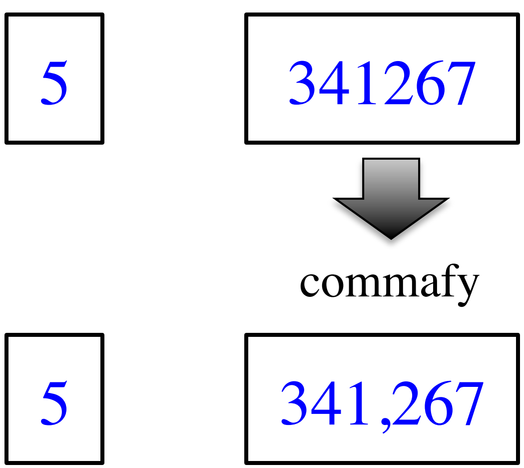

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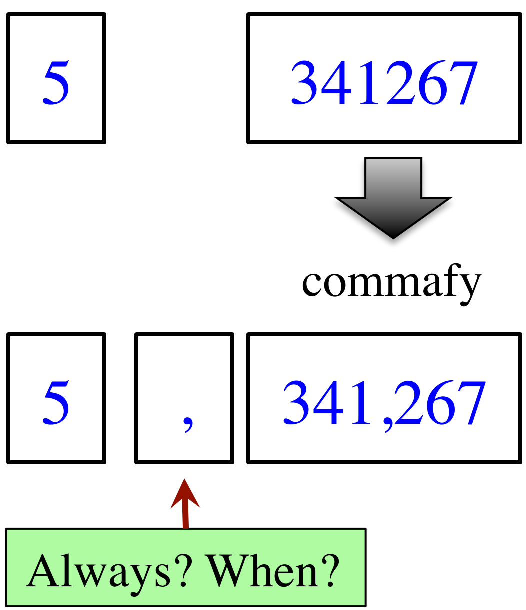

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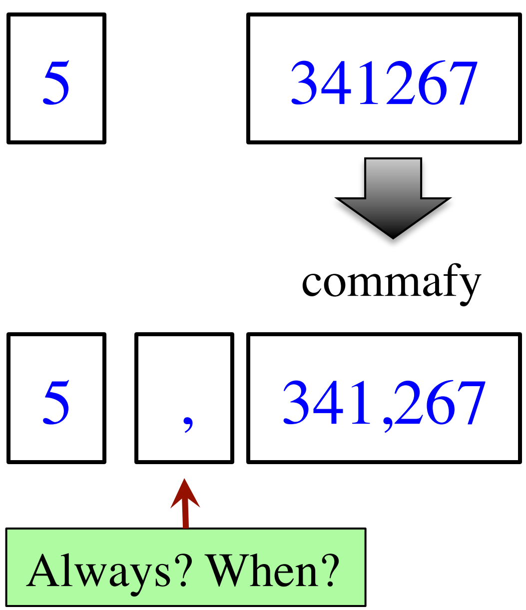
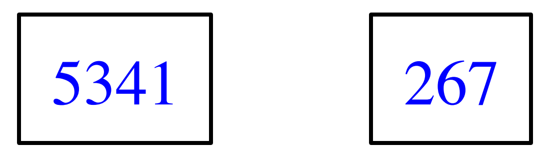


Approach 2



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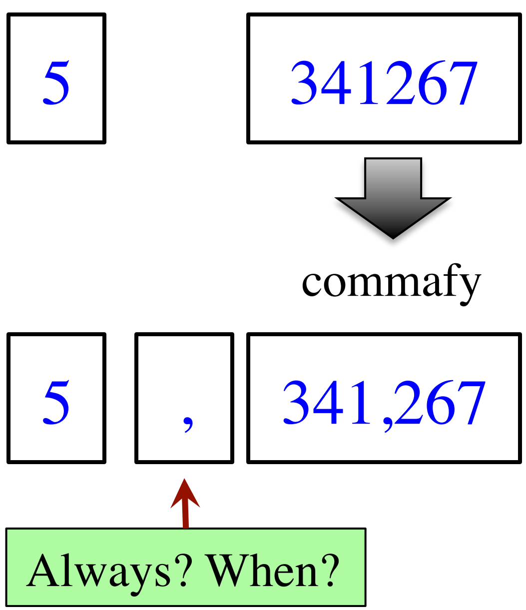
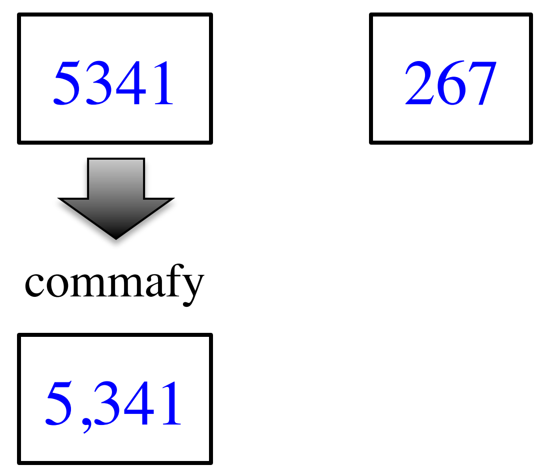


Approach 2



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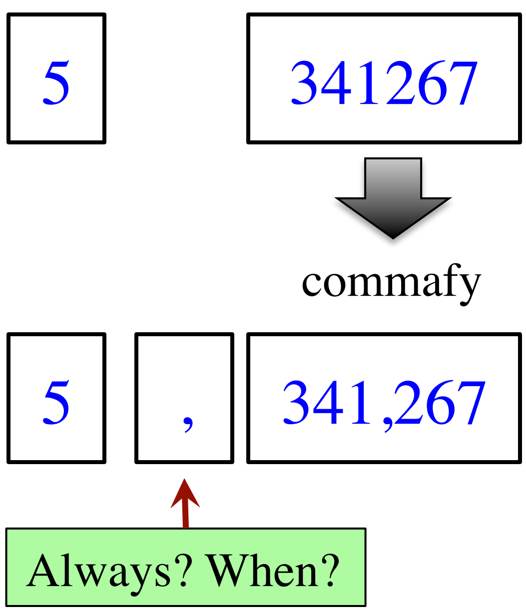

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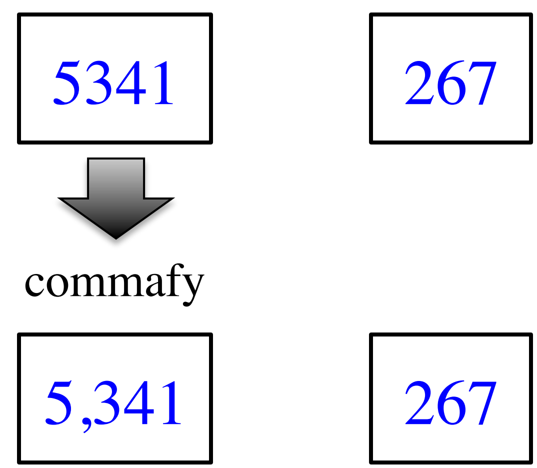

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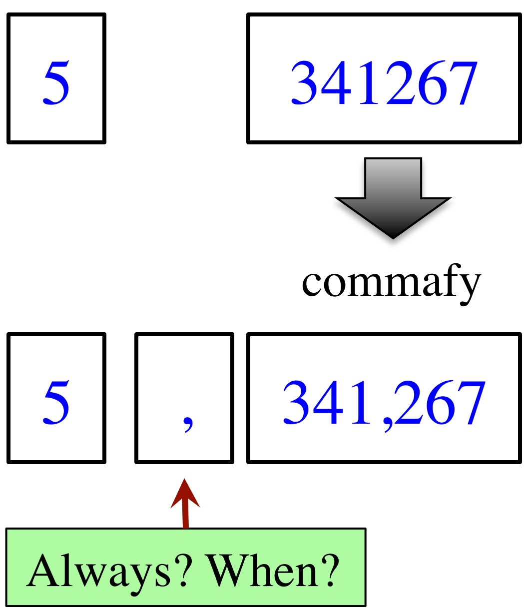

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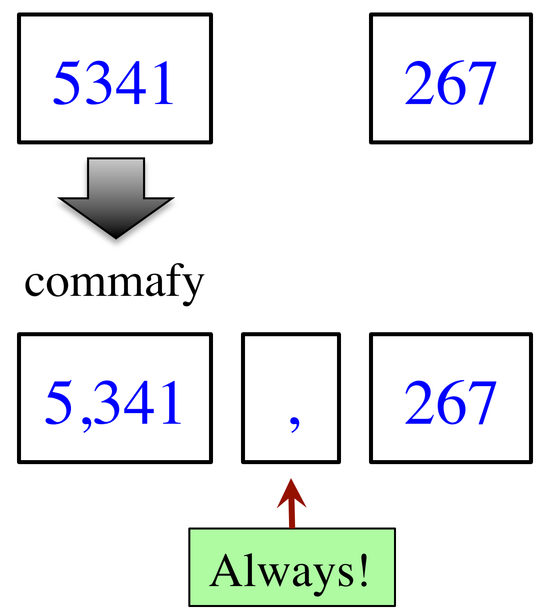

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    """Returns: string with commas every 3 digits
    e.g. commafy('5341267') = '5,341,267'
    Precondition: s represents a non-negative int"""
    # No commas if too few digits.
    if len(s) <= 3:
        return s

    # Add the comma before last 3 digits
    return commafy(s[:-3]) + ',' + s[-3:]
```

Base case

Recursive case

How to Break Up a Recursive Function?

```
def exp(b, c)
    """Returns: b^c
    Precondition: b a float, c ≥ 0 an int"""
```

Approach 1

$$12^{256} = 12 \times (12^{255})$$

Recursive

$$b^{c} = b \times (b^{c-1})$$

Approach 2

$$12^{256} = (12^{128}) \times (12^{128})$$

Recursive Recursive

$$b^{c} = (b \times b)^{c/2} \text{ if c even}$$

Raising a Number to an Exponent

Approach 1

```
def exp(b, c)
    """Returns: b^c
    Precondition: b a float,
                  c ≥ 0 an int"""
    # b^0 is 1
    if c == 0:
        return 1

    # b^c = b(b^c)
    return b*exp(b,c-1)
```

Approach 2

```
def exp(b, c)
    """Returns: b^c
    Precondition: b a float,
                  c ≥ 0 an int"""
    if c == 0:
        return 1
    # c > 0
    if c % 2 == 0:
        return exp(b*b,c/2)

    return b*exp(b*b,c/2)
```

Raising a Number to an Exponent

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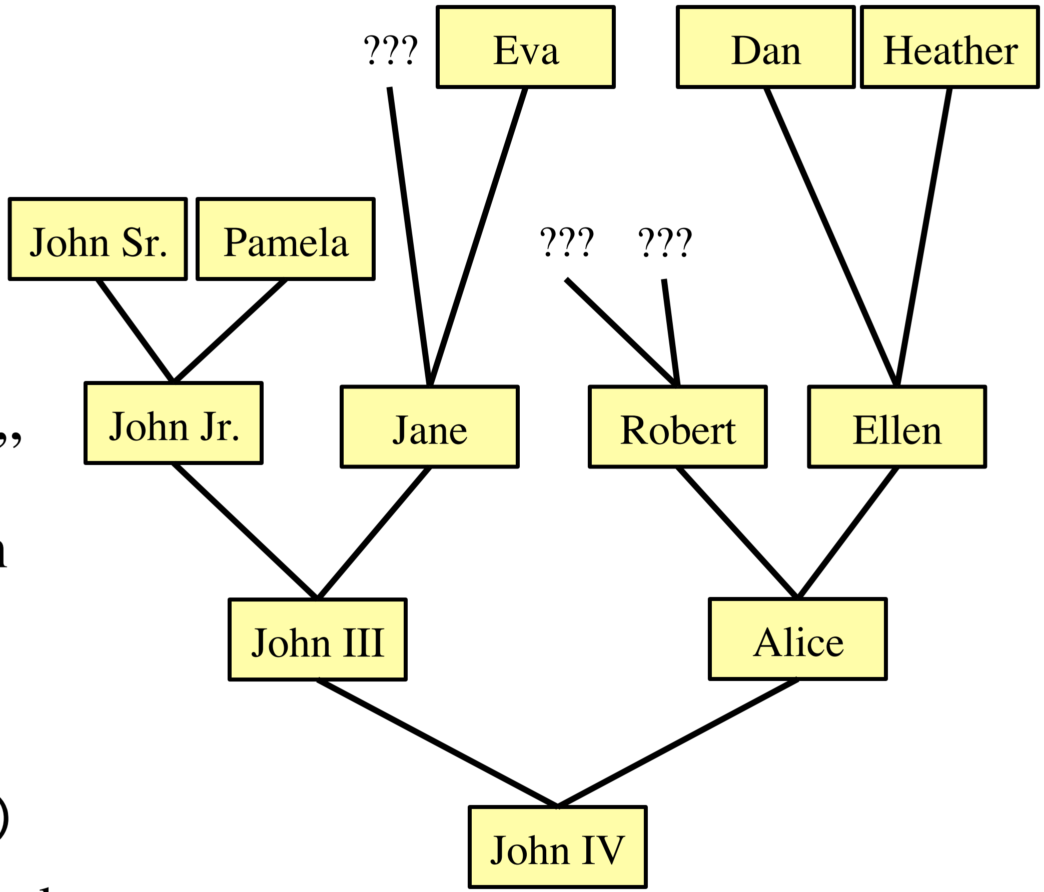
    return b*exp(b*b,c/2)
```

c	# of calls
0	0
1	1
2	2
4	3
8	4
16	5
32	6
2^n	n + 1

32768 is 215
b^{32768} needs only 215 calls!

Recursion and Objects

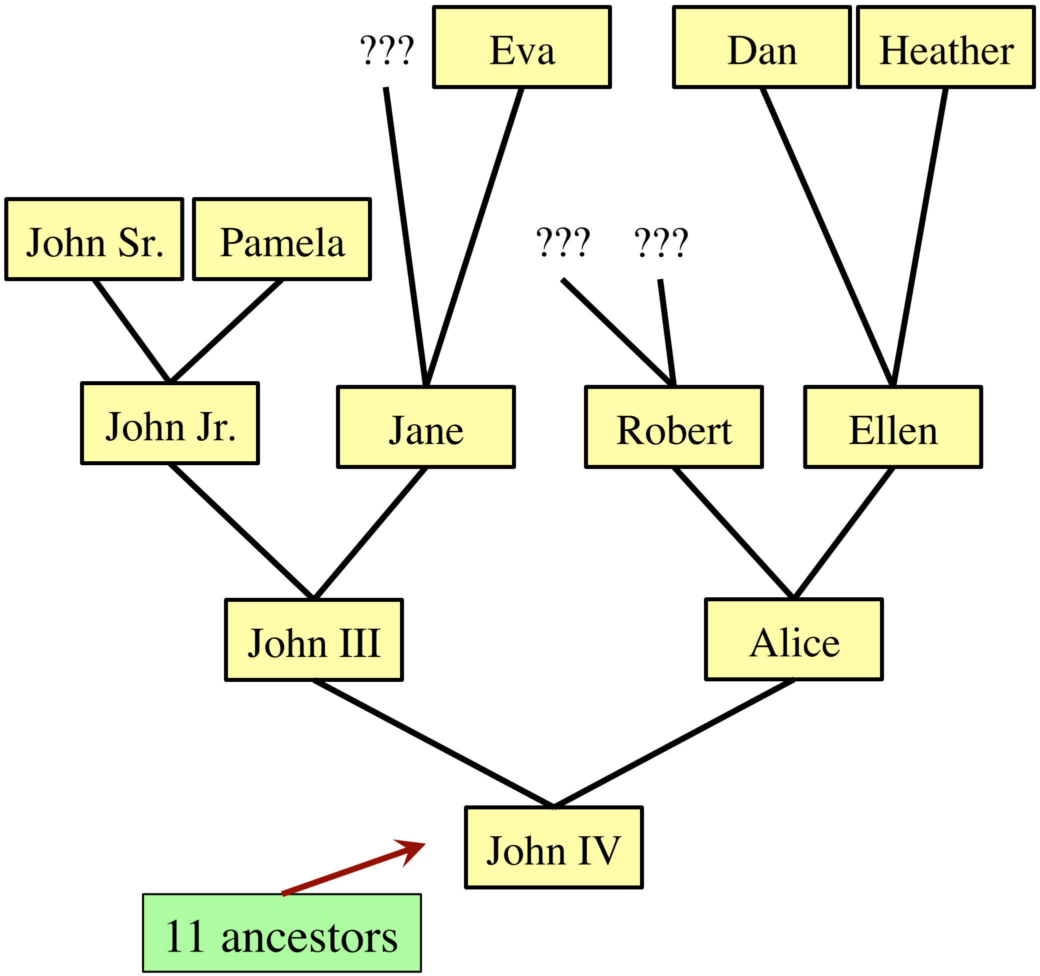
- Class Person (`person.py`)
  - Objects have 3 attributes
  - `name`: String
  - `mom`: Person (or `None`)
  - `dad`: Person (or `None`)
- Represents the "family tree"
  - Goes as far back as known
  - Attributes `mom` and `dad` are `None` if not known
- **Constructor**: `Person(n,m,d)`
  - Or `Person(n)` if no mom, dad



Recursion and Objects

```
def num_ancestors(p):
    """Returns: num of known ancestors
    Pre: p is a Person"""
    # Base case
    # No mom or dad (no ancestors)

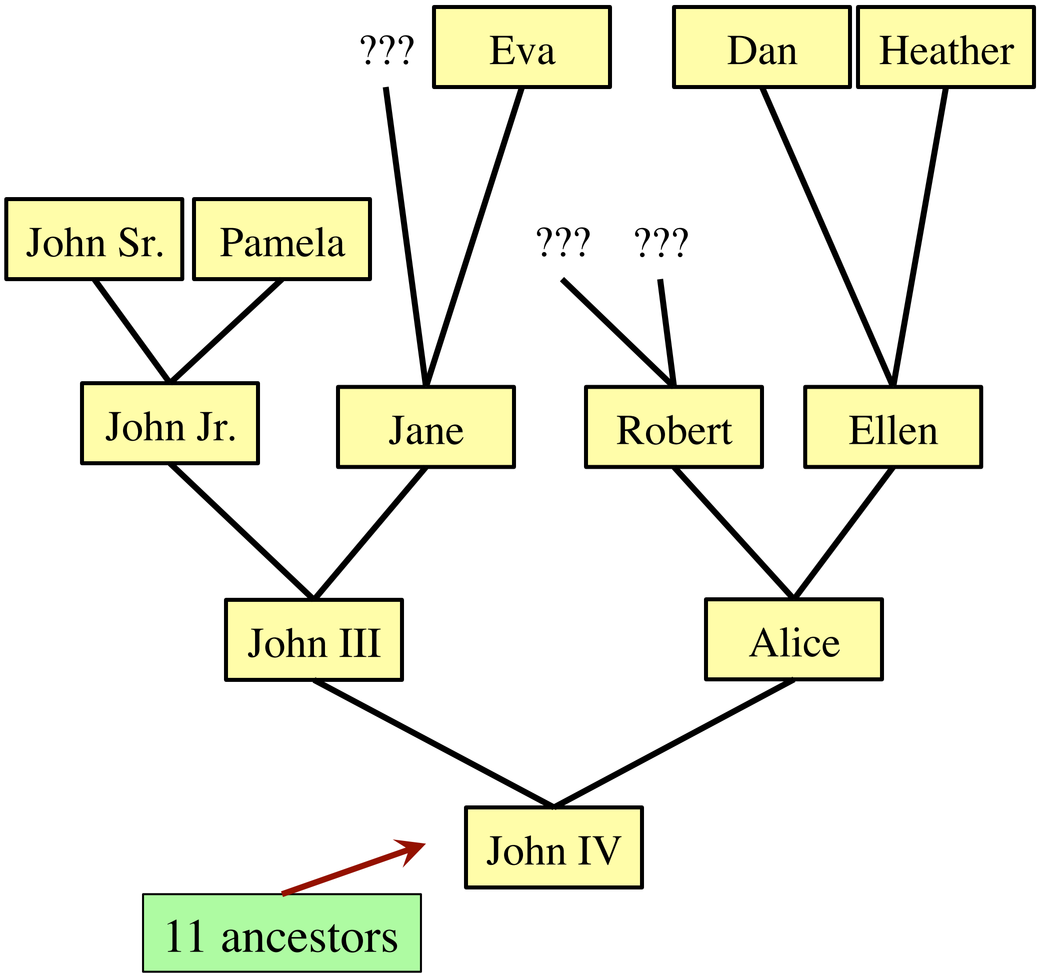

    # Recursive step
    # Has mom or dad
    # Count ancestors of each one
    # (plus mom, dad themselves)
    # Add them together
```



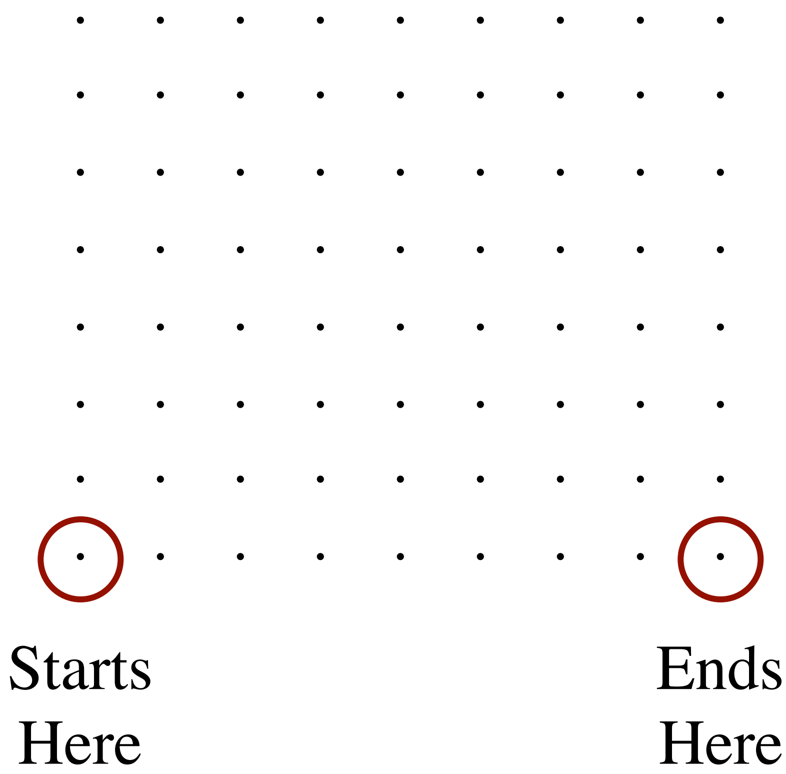
Recursion and Objects

```
def num_ancestors(p):
    """Returns: num of known ancestors
    Pre: p is a Person"""
    # Base case
    if p.mom == None and p.dad == None:
        return 0

    # Recursive step
    moms = 0
    if not p.mom == None:
        moms = 1+num_ancestors(p.mom)
    dads = 0
    if not p.dad== None:
        dads = 1+num_ancestors(p.dad)
    return moms+dads
```



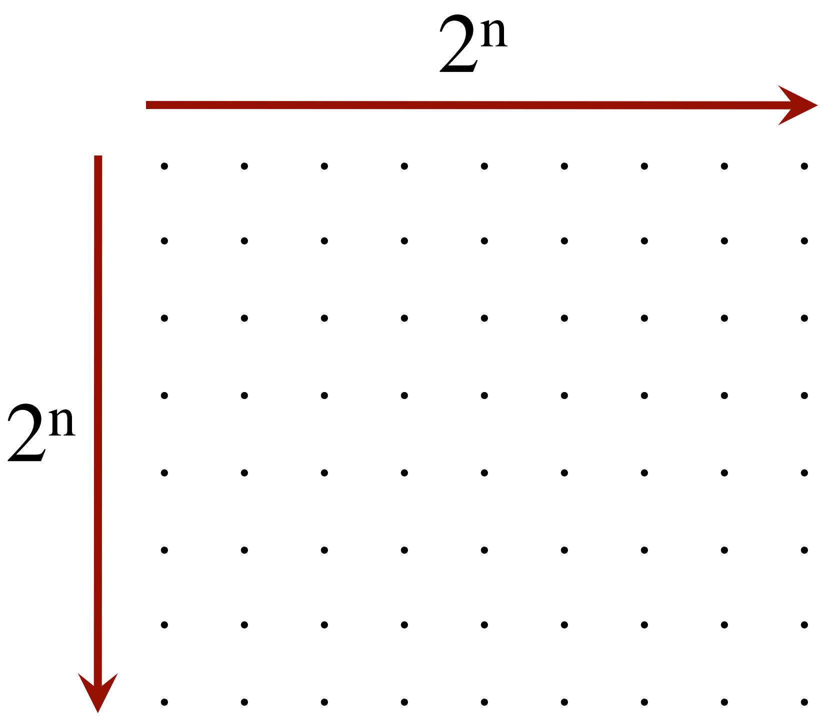
Space Filling Curves

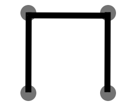


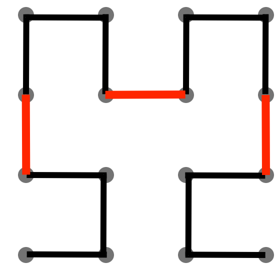
Challenge

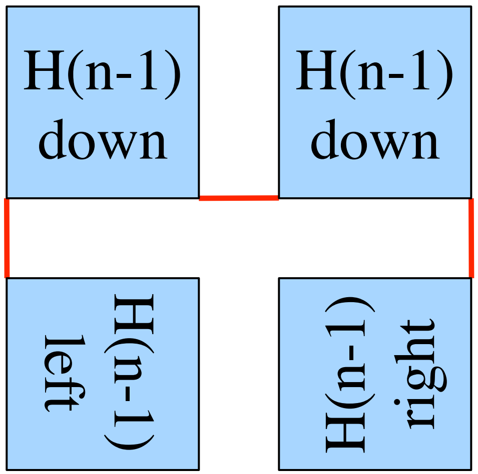
- Draw a curve that
  - Starts in the left corner
  - Ends in the right corner
  - Touches every grid point
  - Does not touch or cross itself anywhere
- Useful for analysis of 2-dimensional data

Hilbert's Space Filling Curve



Hilbert(1): 

Hilbert(2): 

Hilbert(n): 

Hilbert's Space Filling Curve

Basic Idea

- Given a box
- Draw $2^n \times 2^n$ grid in box
- Trace the curve
- As n goes to ∞, curve fills box