

Recursion

- Recursive Definition:**
A definition that is defined in terms of itself
- Recursive Function:**
A function that calls itself (directly or indirectly)

- Recursion:** If you get the point, stop; otherwise, see Recursion
- Infinite Recursion:** See Infinite Recursion

A Mathematical Example: Factorial

- Non-recursive definition:
 $n! = n \times n-1 \times \dots \times 2 \times 1$
 $= n (n-1 \times \dots \times 2 \times 1)$
- Recursive definition:
 $n! = n (n-1)! \quad \text{for } n \geq 0$ **Recursive case**
 $0! = 1$ **Base case**

What happens if there is no base case?

Example: Fibonacci Sequence

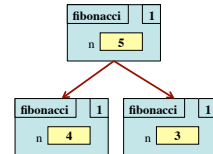
- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...
 $a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6$
 - Get the next number by adding previous two
 - What is a_8 ?
- Recursive definition:
 - $a_n = a_{n-1} + a_{n-2}$ **Recursive Case**
 - $a_0 = 1$ **Base Case**
 - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

Fibonacci as a Recursive Function

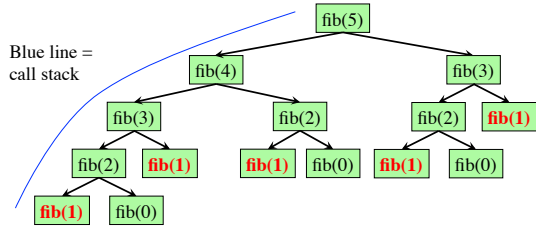
```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1
    return fibonacci(n-1)+
           fibonacci(n-2))
```

- Function that calls itself
 - Each call is new frame
 - Frames require memory
 - ∞ calls = ∞ memory



Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
 - $fib(n)$ has a stack that is always $\leq n$
 - But $fib(n)$ makes a lot of **redundant calls**



String: Two Recursive Examples

```
def length(s):
    """Returns: # chars in s"""
    # { s is empty }
    if s == "":
        return 0
    # { s at least one char }
    return 1 + length(s[1:])

def num_es(s):
    """Returns: # of 'e's in s"""
    # { s is empty }
    if s == "":
        return 0
    # { s at least one char }
    return ((1 if s[0] == 'e'
            else 0) +
            num_es(s[1:]))
```

Imagine len(s) does not exist

How to Think About Recursive Functions

- 1. Have a precise function specification.**
- 2. Base case(s):**
 - When the parameter values are as small as possible
 - When the answer is determined with little calculation.
- 3. Recursive case(s):**
 - Recursive calls are used.
 - Verify recursive cases with the specification
- 4. Termination:**
 - Arguments of calls must somehow get "smaller"
 - Each recursive call must get closer to a base case

Understanding the String Example

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # { s is empty }
    if s == "":
        return 0
    # { s at least one char }
    return ((1 if s[0] == 'e' else 0)
            + num_es(s[1:]))
```

0 1 len(s)
s H ello World!

- Break problem into parts
 - number of e's in s =
number of e's in s[0] + number of e's in s[1:]
- Solve small part directly
 - number of e's in s =
(1 if s[0] = 'e' else 0) + number of e's in s[1:]

Understanding the String Example

- Step 1: Have a precise specification**

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # { s is empty }
    if s == "":
        return 0
    # { s at least one char }
    # return # of 'e's in s[0]+# of 'e's in s[1:]
    return (1 if s[0] == 'e' else 0) + num_es(s[1:])
```

“Write” your return statement using the specification

Recursive case

- Step 2: Check the base case**
 - When s is the empty string, 0 is returned.
 - So the base case is handled correctly.

Understanding the String Example

- Step 3: Recursive calls make progress toward termination**

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # { s is empty }
    if s == "":
        return 0
    # { s at least one char }
    # return # of 'e's in s[0]+# of 'e's in s[1:]
    return (1 if s[0] == 'e' else 0) + num_es(s[1:])
```

parameter s

argument s[1:] is smaller than parameter s, so there is progress toward reaching base case 0

argument s[1:]

- Step 4: Recursive case is correct**
 - Just check the specification

Exercise: Remove Blanks from a String

- 1. Have a precise specification**

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

- 2. Base Case:** the smallest String s is "".

```
if s == "":
    return s
```

- 3. Other Cases:** String s has at least 1 character.

```
return (s[0] with blanks removed) + (s[1:] with blanks removed)
```

(" if s[0] == ' ' else s[0])

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s with blanks removed"""
    if s == "":
        return s
    # s is not empty
    if s[0] is a blank:
        return s[1:] with blanks removed
    # s not empty and s[0] not blank
    return (s[0] +
           s[1:] with blanks removed)
```

- Sometimes easier to break up the recursive case
 - Particularly on small part
 - Write recursive case as a sequence of if-statements
- Write code in *pseudocode*
 - Mixture of English and code
 - Similar to top-down design
- Stuff in **red** looks like the function specification!
 - But on a smaller string
 - Replace with deblank(s[1:])