SYSTEMS OF EQUATIONS

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Adapted from *Introduction to Maple*, by David I. Schwartz, Prentice Hall, 1999.

Example Problem

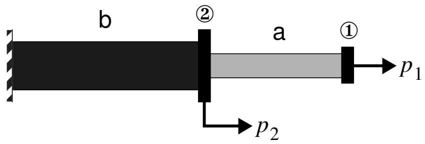


Figure 0-1: Example Device

A device composed of two elastic bars capped with rigid plates is shown in Fig. 0-1. Loads are statically applied on both plate ① and plate ②. For now, call the connections at ① and ② as *nodes*.

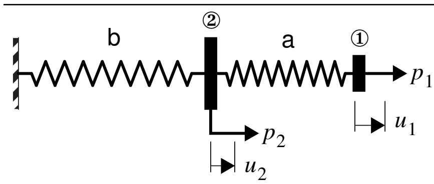
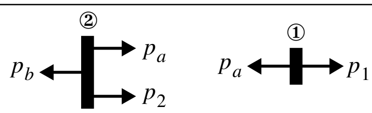


Figure 0-2: Model of Device

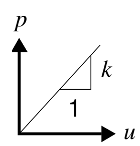
You can model elastic bars as springs as shown in Fig. 0-2. Assume that Hooke's law $p = ku$ governs spring behavior as discussed in Chapter 6.



$$p_b = p_2 + p_a = p_1 + p_2 \qquad p_a = p_1$$

Figure 0-3: Free Body Diagrams

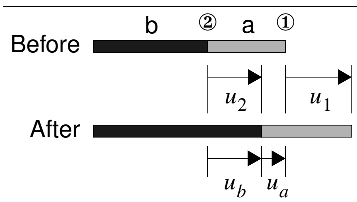
After slowly applying loads, the bars deform and reach a new resting or *equilibrium* position. The applied loads and internal bar forces must balance according to equilibrium. Assume no twisting or rotation of the plates occur.



$$p_a = k_a u_a$$
$$p_b = k_b u_b$$

Figure 0-4: Hooke's Law

From Hooke's Law, relate each spring's internal force with to *relative displacement*, the measure of how much each spring stretches. Bars a and b have relative displacements u_a and u_b, respectively.



$$u_b = u_2 \qquad u_1 = u_2 + u_a$$
$$\therefore u_a = u_1 - u_2$$

Figure 0-5: Relative Displacements

Displacement at ② stretches spring b and compresses spring a. Displacement at ① (u_1) stretches spring a. Therefore, determine a's "relative stretch" by subtracting node ②'s displacement (u_2) from that of node ① as shown in Fig. 0-5.

$$\begin{aligned} p_1 &= p_a \\ &= k_a u_a = k_a(u_1 - u_2) \\ p_2 &= p_b - p_1 \\ &= k_b u_2 - k_a(u_1 - u_2) \\ &= -k_a u_1 + (k_a + k_b) u_2 \end{aligned}$$

Figure 0-6: Combine Equations

Ease your computation by expressing the equations in terms of nodal values as shown in Fig. 0-6. Combine equilibrium, Hooke's Law, and displacement relations into two equations in terms of p, k, and u.

Systems of Equations

Rearrange the equations of Fig. 0-6 into the ***system of equations***,

$$k_a u_1 - k_a u_2 = p_1 \tag{1}$$

$$-k_a u_1 + (k_a + k_b) u_2 = p_2 \tag{2}$$

A system of equations collects simultaneous equations with common unknown variables also called *unknowns* or *indeterminates*. The system in Eq. 1 and 2 has two unknowns, u_1 and u_2. Assume all other variables have predetermined values.

Linear systems contain all first-order equations in the form $a_0 = a_1 x_1 + \ldots + a_n x_n$. Both Eq. 1 and 2 contain terms with powers no greater than one, and thus, constitute a linear system. On the other hand, ***non-linear systems*** of equations contain terms with powers higher than one.

Gaussian Elimination

Solve for the unknowns u_1 and u_2 as shown in Fig. . First, assume values $k_a = 2$, $k_b = 3$, $p_1 = 10$, and $p_2 = 20$ in Step ①. Apply ***gaussian elimination*** as demonstrated by Steps ②, ③, and ④. By adjusting coefficients, you can eliminate common terms. After dividing out leading coefficients, solve for unknowns by backsubstitution.

① Substitute values into Eqs. 1 and 2.	② Reduce the second equation by adding the first.	③ Divide equations by the first or leading coefficient.	④ Backsubstitute results into the first equation.
$2u_1 - 2u_2 = 10$ $-2u_1 + 5u_2 = 20$	$2u_1 - 2u_2 = 10$ $3u_2 = 30$	$u_1 - u_2 = 5$ $u_2 = 10$	$u_1 = 15$ $u_2 = 10$

Figure 1: Gaussian Elimination

Dependency

The following conditions characterize a ***linearly independent*** system of equations:

- The number of equations match the number of unknowns.
- No equation is a multiple of another equation in the system.

A linearly independent system produces only one, or unique, solution for each unknown as demonstrated in Figure 1. You might encounter systems with duplicates of equations, such as the system $x + y = 1$ and $2x + 2y = 2$. Such a system produces an infinite number of x and y solutions and is called ***linearly dependent***.

Denote the resulting "matrix form" of the equations as

$$Ku = p \tag{3}$$

where matrix multiplication and equality are implied. Each term is described below:

System $Ku = p$	The ***system of equations***, $Ku = p$, is a set of simultaneous linear equations.
Coefficient Matrix $K = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$	The ***coefficient matrix***, K, collects the constants in front of unknowns: • Square matrices, K, have same the number of unknowns and equations. • Elements of these matrices typically reflect models' physical parameters.
Source Vector $p = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$	The ***source vector***, p, applies modeled inputs or "sources" to the system: • In the spring example, these source terms are loads. • Source values are typically known or assumed.
Solution Vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$	The ***solution vector***, u, collects the unknown variables you wish to find: • The matrix formulation separates known and unknown variables. • Manipulating the coefficient matrix and source vector with linear algebra techniques like Gaussian elimination (Fig.) finds the unknowns.

MANUAL SOLUTION

As demonstrated in this document, you can perform Gaussian elimination to solve the system of equations that Eq. 3 represents. But, why should you use the matrix formulation? Vectors and matrices store equation data in a compact form that computer programs can readily manipulate. Though more complex techniques exist, you can solve the matrix formulation with ***row reduction***, which is a method that mimics Gaussian elimination as demonstrated in Table 2.

The following steps illustrate row reduction:

- Step ①: Cast the equations into a matrix formulation.
- Step ②: Rewrite the system into a matrix that includes the source vector written to the right. You may draw a vertical bar to serve as a reminder to separate the coefficient matrix.
- Step ③: Row reduction dictates that you may add a row to any other row. So, add the top row to the bottom row. This process is equivalent to adding an equation to another equation within the given system.
- Step ④: Row reduction also dictates you can multiply any row by any constant. So, divide the top row by 2, and divide the bottom row by 3. Note that you can perform this action in conjunction with adding rows to each other.
- Step ⑤: You keep performing row reduction until the coefficient matrix becomes the *identity matrix*, a matrix with values of 1 on the diagonal and 0 elsewhere. The final values in the right-hand column of the matrix represent the solution vector.

Note that a linearly dependent system of equations will yield at least one row that contains only values of zero.

TABLE 2: ROW REDUCTION

Step	Matrix Formulation	Operations	Results
①	$\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$	$\begin{aligned} 2u_1 - 2u_2 &= 10 \\ -2u_1 + 5u_2 &= 20 \end{aligned}$	$\begin{aligned} 2u_1 - 2u_2 &= 10 \\ -2u_1 + 5u_2 &= 20 \end{aligned}$
②	$\left[\begin{array}{cc\|c} 2 & -2 & 10 \\ -2 & 5 & 20 \end{array}\right]$		
③	$\left[\begin{array}{cc\|c} 2 & -2 & 10 \\ 0 & 3 & 30 \end{array}\right]$	$\begin{array}{rr} & 2u_1 - 2u_2 = 10 \\ + & -2u_1 + 5u_2 = 20 \\ \hline & 0u_1 + 3u_2 = 30 \end{array}$	$\begin{aligned} 2u_1 - 2u_2 &= 10 \\ 3u_2 &= 30 \end{aligned}$
④	$\left[\begin{array}{cc\|c} 1 & -1 & 5 \\ 0 & 1 & 10 \end{array}\right]$	$\frac{1}{2}(2u_1 - 2u_2 = 10) \rightarrow u_1 - u_2 = 5$ $\frac{1}{3}(0u_1 + 3u_2 = 30) \rightarrow u_2 = 10$	$\begin{aligned} u_1 - u_2 &= 5 \\ u_2 &= 10 \end{aligned}$
⑤	$\left[\begin{array}{cc\|c} 1 & 0 & 15 \\ 0 & 1 & 10 \end{array}\right]$	$\begin{array}{rr} & u_1 - u_2 = 5 \\ + & u_2 = 10 \\ \hline & u_1 + 0u_2 = 15 \end{array}$	$\begin{aligned} u_1 &= 15 \\ u_2 &= 10 \end{aligned}$