Exercises on proofs of $f(x) \in O(g(n))$

Our proofs of these theorems are given at the end of this document. There are other ways to prove these theorems; the constants $c$ and $N$ that we come up with are not the only possibilities. We prefer our proof format for two reasons: (1) with each step, it explains why the step is logical. (2) Each proof has a natural progression, changing $f(n)$ into $c \cdot g(n)$.

**Theorem 1.** $20n \in O(n)$.

**Theorem 2.** $n \in O(20n)$.

**Theorem 3.** $n + 50 \in O(n)$.

**Theorem 4.** $\log n \in O(n)$.

**Theorem 5.** If $f(n) \in O(n)$ and $h(n) \in O(n)$ then $f(n) + g(n) \in O(n)$.

That is, if two functions are in $O(n)$, so is their sum.

**Theorem 6.** $n + \log n \in O(n)$.

**Theorem 7.** If $f(n) \in O(g(n))$ and $h(n) \in O(g(n))$ then $f(n) + g(n) \in O(g(n))$.

**Theorem 8.** $5n^2 + 50n \in O(n^2)$.

**Theorem 9.** $n (n-1)/2 \in O(n^2)$.

**Theorem 10.** $80 \cdot 2^n + 60n^3 \in O(2^n)$.

**Definition.** Function $f(n) \in O(g(n))$ iff there exist positive constants $c$ and $N$ such that for all $n$, $n \geq N$, $f(n) \leq c \cdot g(n)$. 
Exercises on proofs of $f(x) \in O(g(n))$

**Theorem 1.** $20n \in O(n)$. We change $20n$ into $c\,n$, finding $N$ and $c$ as we go.

$$20n$$

$$= \quad \text{Choose } N = 1 \text{ and } c = 20$$

$$c \, n \quad \text{for } n \geq N$$

**Theorem 2.** $n \in O(20n)$. We change $n$ into $c(20n)$, finding $N$ and $c$ as we go.

$$n$$

$$< \quad \text{Choose } N = 1. \text{ For all } n \geq 1, n < 2n$$

$$20 \, n \quad \text{for } n \geq N$$

$$< \quad \text{Choose } c = 1$$

$$c \, (20n) \quad \text{for } n \geq N$$

**Theorem 3.** $n + 50 \in O(n)$. We change $n + 50$ into $c\,(20n)$, finding $N$ and $c$ as we go.

$$n + 50$$

$$\leq \quad \text{Choose } N = 50. \text{ For all } n \geq 50, 50 \leq n$$

$$n + n \quad \text{for } n \geq N$$

$$< \quad \text{Arithmetic, and choose } c = 2$$

$$c \, n \quad \text{for } n \geq N$$

**Theorem 4.** $\log n \in O(n)$. We change $\log n$ into $c\, n$, finding $N$ and $c$ as we go.

$$\log n$$

$$\leq \quad \text{Choose } N = 1. \text{ If } n = 2^y, \log n = y. \text{ From this, we know that for } n \geq 1, \log n \leq n$$

$$n \quad \text{for } n \geq N$$

$$< \quad \text{Choose } c = 1$$

$$c \, n \quad \text{for } n \geq N$$

**Theorem 5.** If $f(n) \in O(n)$ and $h(n) \in O(n)$ then $f(n) + h(n) \in O(n)$.

**Proof.** Since $f(n) \in O(n)$, there exist positive $N1$ and $c1$ such that $f(n) \leq c1 \, n$ for all $n \geq N1$.

Since $h(n) \in O(n)$, there exist positive $N2$ and $c2$ such that $h(n) \leq c2 \, n$ for all $n \geq N2$. We calculate:

$$f(n) + h(n)$$

$$\leq \quad \text{Above-mentioned fact about } f(n) \in O(n)$$

$$c1 \, n + h(n) \quad \text{for } n \geq N1$$

$$\leq \quad \text{Above-mentioned fact about } h(n) \in O(n)$$

$$c1 \, n + c2 \, n \quad \text{for } n \geq N1 \text{ and } n \geq N2$$

$$\leq \quad \text{Choose } N = \max(N1, N2)$$

$$c1 \, n + c2 \, n \quad \text{for } n \geq N$$

$$\leq \quad \text{Arithmetic, and choose } c = c1 + c2$$

$$c \, n \quad \text{for } n \geq N$$

**Theorem 6.** $n + \log n \in O(n)$.

**Proof.** You can prove this as we did most of the previous ones. However, you can also apply theorem 5, since by theorems 1 and 2, $n \in O(n)$, and by theorem 4, $\log n \in O(n)$.

**Theorem 7.** If $f(n) \in O(g(n))$ and $h(n) \in O(g(n))$ then $f(n) + g(n) \in O(g(n))$.

**Proof.** This proof will be exactly the same as the proof of Theorem 5, with all occurrences of $O(n)$ replaced by $O(g(n))$.

**Theorem 8.** $5n^2 + 50n \in O(n^2)$. We change $5n^2 + 50n$ into $c\,n^2$, finding $N$ and $c$ as we go.

$$5n^2 + 50n$$

$$\leq \quad \text{Choose } N = 1, \text{ because } 50 \, n \leq 50n^2 \text{ for } n \geq 1$$
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\[5n^2 + 50n^2 \quad \text{for } n \geq N\]
\[\leq \quad <\text{Arithmetic}>\]
\[55n^2 \quad \text{for } n \geq N\]
\[\leq \quad <\text{Choose } c = 55>\]
\[cn^2 \quad \text{for } n \geq N\]

**Theorem 9.** $n(n-1)/2 \in O(n^2)$. We change $5n^2 + 50n$ into $c \cdot n^2$, finding $N$ and $c$ as we go.

\[n(n-1)/2\]
\[\leq \quad <\text{Arithmetic}>\]
\[n^2/2 - n/2\]
\[\leq \quad <\text{Arithmetic}>\]
\[n^2\]
\[\leq \quad <\text{Choose } N = 1 \text{ and } N = 1>\]
\[cn^2 \quad \text{for } n \geq N\]

**Theorem 10.** $80 \cdot 2^n + 60n^3 \in O(2^n)$. We change $80 \cdot 2^n + 60n^3$ into $c \cdot 2^n$, finding $N$ and $c$ as we go.

\[80 \cdot 2^n + 60n^3\]
\[\leq \quad <\text{For } n = 10, n^3 = 1000 < 1024 = 2^n. \text{ For } n > 10, 2^n > n^3. \text{ Choose } N = 10.>\]
\[80 \cdot 2^n + 60 \cdot 2^n \quad \text{for } n \geq N\]
\[\leq \quad <\text{Arithmetic}>\]
\[140 \cdot 2^n \quad \text{for } n \geq N\]
\[\leq \quad <\text{Choose } c = 140>\]
\[c \cdot 2^n \quad \text{for } n \geq N\]