The identity of an operation

The box to the right defines three variables, \( b \), \( s \), and \( p \). For example, if \( b \) contains the values \( \{2, 3, 2\} \), \( s = 7 \), and \( p = 12 \).

Suppose \( b \) is empty. Then \( s = 0 \) and \( p = 1 \). The big question then arises: why is the product of an empty bag 1? This little note answers that question and talks about the identity of an operation.

**Why 1 is the product of an empty bag:**

Suppose \( b \) contains \( \{2, 3, 2\} \), so \( s = 7 \) and \( p = 12 \). Suppose we want to insert the value 5 in \( b \) but keep all definitions true, we would execute:

\[
\begin{align*}
\text{Add 5 to } b; \\
\text{s} &= s + 5; \\
\text{p} &= p * 5;
\end{align*}
\]

That same sequence should be used to add 5 to \( b \) no matter what \( b \) is. Suppose bag \( b \) is empty and we add 5 to it. Therefore, we want to set \( p \) to 5. What value should be in \( p \) when the bag is empty to that execution of

\[
\text{p} = p * 5;
\]

sets \( p \) to 5? Obviously, \( p \) must be 1. That is why the product of an empty bag is defined to be 1.

**The identity of a binary operation**

We define the identity of common operators. The generalization to any binary operator is clear, although not all binary operators have identities:

- 0 is the identity of + because \( 0 + x = x \)
- 1 is the identity of * because \( 1 + x = x \)
- false is the identity of || because \( \text{false} || c = c \)
- true is the identity of && because \( \text{true} && c = c \)

For any binary operator \( o \) with an identity, \( o \) applied to the empty bag is the identity of \( o \).

- The sum of an empty bag is 0.
- The product of an empty bag is 1.
- The disjunction (“or” ||) of an empty bag is false
- The conjunction (“and” &&) of an empty bag is true.