Examples of Smooth Auctions (Part 1)

Scribe: Jiayang Gao

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Instructor: Eva Tardos

Last lecture, we defined smoothness of auctions as following:

Definition 1. *An auction game is* (λ, μ) *smooth if* $\exists s^*$, *s.t,* $\sum_i u_i(s_i^*, s_{-i}) \geq \lambda OPT - \mu \sum_i p_i(s)$. *Where* $o(s)$ *is the outcome at strategy vector* s, $V_i(o(s))$ *is the value of player* i *at outcome* $o(s)$, $p_i(s)$ *is the payment of player* i *given strategy vector* s, *and* $u_i(s) = V_i(o(s)) - p_i(s)$, $OPT = \max_o \sum_i V_i(o)$.

Using smoothness, we also had the following two theorems on PoA bounds for full info. game and Bayesian game (respectively).

Theorem 1. *For a full information game,* (λ, μ) *smooth implies for any Nash* s, $SW(s) \geq \frac{\lambda}{\max(1,\mu)} OPT$.

Theorem 2. *For a Bayesian game,* (λ, μ) *smooth with* s_i^* *depends only on* v_i *for all* i, *implies for any Nash* s, $E[SW(s)] \geq \frac{\lambda}{\max(1,\mu)} E[OPT]$.

In this lecture and next lecture, we will look at examples of smooth games.

Example 1: First Price Auction of a single item

- Players $1, \ldots, n$.
- Values of getting the item $(v_1, \ldots, v_n)$, and value = 0 if not getting it.
- Bids $(b_1, \ldots, b_n)$.

We use the following simple argument to show that the game is $(\frac{1}{2}, 1)$ smooth if we let $s_i^* = \frac{v_i}{2}$ for all i.

Proof. If $j = \arg\max_i v_i$, then $u_j(s_j^*, s_{-j}) \geq \frac{1}{2} v_j - \sum_i p_i(s)$ because

- If j wins, $u_j = v_j - s_j^*(v_j) = \frac{v_j}{2} \geq \frac{1}{2} v_j - \sum_i p_i(s)$.
- If j loses, $u_j = 0$, and $\max_i b_i > \frac{1}{2} v_j$. Notice that $\sum_i p_i(s) = \max_i b_i$ because the maximum bid person pays his bid, and others pays 0. Therefore, $u_j = 0 > \frac{1}{2} v_j - \sum_i p_i(s)$.

If $i \neq \arg\max_i v_i$, then $u_i(s_i^*, s_{-i}) \geq 0$ because if wins, utility is half of his value which is positive, and if loses, utility is 0.

Sum up over all players we get

$$\sum_i u_i(s_i^*, s_{-i}) \geq \frac{1}{2}v_j - \sum_i p_i(s) = \frac{1}{2}OPT - \sum_i p_i(s)$$

Thus the game is $(\frac{1}{2}, 1)$ smooth. □

Thus, according to Theorem 1 and Theorem 2, (notice Theorem 2 applies because here s_i^* only depends on v_i), we have $SW(s) \geq \frac{1}{2}OPT$ for full info game and $E[SW(s)] \geq \frac{1}{2}E[OPT]$ for Bayesian game.

In fact, we can get a tighter bound on PoA as follows.

Theorem 3. *For the single item first price auction defined above, the game is* $(1 - \frac{1}{e}, 1)$ *smooth.*

Proof. Let b_i be randomly chosen according to probability distribution $f(x) = \frac{1}{v_i - x}$ from the interval $[0, (1 - \frac{1}{e}v_i)]$. This probability distribution is well defined because $\int_0^{v_i(1-\frac{1}{e})} \frac{1}{v_i - x}dx = [-\ln(v_i - x)]_0^{v_i(1-\frac{1}{e})} = -\ln(\frac{v_i}{e}) + \ln(v_i) = \ln(\frac{v_i}{v_i/e}) = 1$.

We use the similar technique as above, that

- If $i \neq \arg\max_i v_i$, then $u_i(s_i^*, s_{-i}) \geq 0$.
- If $i = \arg\max_i v_i$. Then $v_i = OPT$. Let $p = \max_{j \neq i} b_j$, then $u_j(s_j^*, s_{-j}) = \int_p^{v_i(1-\frac{1}{e})} f(x)(v_i - x)dx = v(1 - \frac{1}{e}) - p = v_i(1 - \frac{1}{e}) - \max_{j \neq i} b_j \geq v_i(1 - \frac{1}{e}) - \max_j b_j = (1 - \frac{1}{e})OPT - \sum_j p_j$.

Sum up over all i we get

$$\sum_i u_i(s_i^*, s_{-i}) \geq (1 - \frac{1}{e})OPT - \sum_i p_i(s)$$

Therefore the game is $(1 - \frac{1}{e}, 1)$ smooth. □

Similarly, according to Theorem 1 and Theorem 2, we have $SW(s) \geq \frac{e-1}{e}OPT$ for full info game and $E[SW(s)] \geq \frac{e-1}{e}E[OPT]$ for Bayesian game.

Comments:

1. For $s_i^* = \frac{v_i}{2}, o(s^*) = OPT$ because bid is monotone in value, so the maximum value player is always getting the item.
2. For s_i^* random in interval $[0, (1 - \frac{1}{e}v_i)]$, it is possible that $o(s^*) \neq OPT$, because there's possibility even for the max value player to bid close to 0. So in this case the max value person not always get the item.
3. So far we analyzed single item auction. We will talk about how to generalize to multiple item auction next time.