CS 6840 – Algorithmic Game Theory (4 pages) Spring 2012

Lecture 10 Scribe Notes

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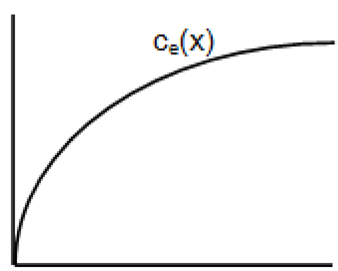
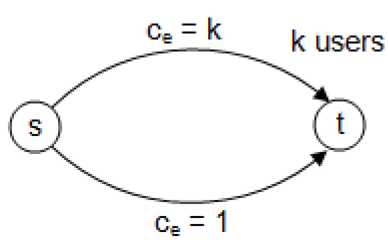
1 Review from last time

We discussed a class of congestion games in which cost decreases with congestion: cost-sharing games. There is a graph G = (V, E), and c_e is cost of edge e. If there are x users on edge e, $cost/user = \frac{c_e}{x}$

More generally: $c_e(x)$ is total cost on edge e for x users, and $cost/user$ is $\frac{c_e(x)}{x}$. In order for the price of anarchy result discussed to work, we need that $c_e(x)$ is monotone increasing and concave (this allows $cost/user$ function $\frac{c_e(x)}{x}$ to be decreasing).



Theorem: $\exists$ a pure Nash s.t. total cost $\leq H_k \cdot \textit{min total cost}$, where $H_k = 1 + \frac{1}{2} + ... + \frac{1}{k}$, k = # of players. I.e, price of stability is $\leq H_k$ (Proof: take potential minimizer).



There are two Nash equilibria in the diagram above.

- Nash 1: share cheap edge, cost 1 total (good Nash)
- Nash 2: share expensive edge, cost k total (bad Nash)

Need to come up with a game-theoretic model that would tell us that intelligent users would not pick Nash 2. We know from our theorem that a good Nash is guaranteed to exist for games such as cost-sharing. Unfortunately, there could also possibly exist bad equilibria. We need a reasonable notion for explaining our intuitive understanding that players would go for the better equilibrium.

2 Allow people to cooperate

Regular Nash Equilibrium : Nash strategy vector s s.t. $\forall$ players i and all other strategies x_i, $cost_i(s) \leq cost_i(x_i, s_{-i})$

Good deviation : Let A represent a group of players. x_i for $i \in A$ is a good deviation from s if $cost_i(x_A, s_{-A}) \leq cost_i(s)$ and strict for at least one $i \in A$. Note: (x_A, s_{-A}) is vector with x_i for $i \in A$, and s_i for $i \notin A$.

Strong Nash Equilibrium : for all groups, there is no good deviation. Or, $\forall$ groups of players A and all other strategies x_i for $i \in A$, either $cost_i(x_A, s_{-A}) > cost_i(s)$ for some $i \in A$, or $=$ for all $i \in A$.

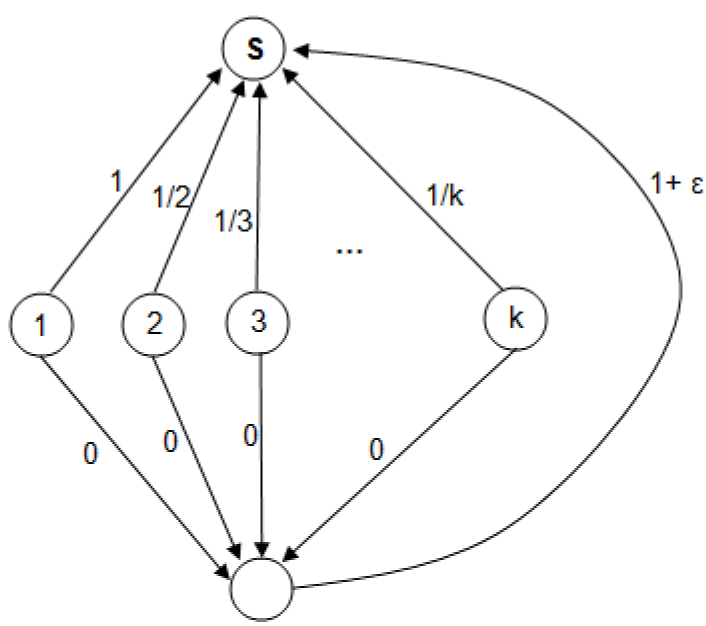
Lemma: every strong Nash is a Nash (consider a group of size one).

In our earlier example, Nash1 is a strong Nash but Nash2 is not. This observation should give us some hope that the idea of strong Nash is what we were looking for.

3 Strong Nash in cost-sharing games

Theorem (Epstein, Feldman, Mansour EC′07) :
All strong Nash in cost-sharing game have cost $\leq H_k \cdot$ *min total cost*.

Example:



Users 1 to k want to connect to node s. Each player i can buy a direct link to s for cost $1/i$, or can join in a shared link of cost $1 + \epsilon$. The optimal solution is for all players to cost-share on the shared link. But, as we saw last class, user k would prefer to switch to his direct path – causing a chain of switches until, in the unique Nash equilibrium, all users take their direct links to s.

This Nash is also a strong Nash, and thus has cost $\leq H_k \cdot$ *min total cost*.

Proof : Let s be a strong Nash. As usual, we will think about why these players are not switching to the opt. We mentioned two reasons why a good deviation may fail: either the utility of all players stays the same or there exists some player whose utility strictly decreases. The first case is not very interesting – if everyone were equally well off after switching to opt then the Nash and opt would cost the same and our theorem must be true. We will consider the case when Nash is more expensive than the opt.

1. We ask all players to switch from s to opt. But $\exists$ player k with $cost_k(s) < cost_k(opt)$
2. We ask players 1, ... , $k-1$ to switch from s to opt (we should actually ask them to switch to their opt, but that's a bit tricky). But $\exists$ player $k-1$ with $cost_{k-1}(s) \leq cost_{k-1}(s_k, opt_{1,\ldots,k-1})$ [the RHS cost expression takes into account the fact that player k is still playing strategy s]

Continuing like this, we can make a general statement. We name the players in the order that comes out of this proof: $\forall i$, $cost_i(s) \leq cost_i(opt_{1,\ldots,i}, s_{i+1,\ldots,k})$. In this expression, person i is the one who currently objects the deviation.

We want to add up the inequalities to come up with an expression relating the total costs of the optimal and strong Nash games.

$$
\begin{aligned}
cost_i(s) &\leq cost_i(opt_{1,\ldots,i}, s_{i+1,\ldots,k}) && (1)\\
&\leq cost_i(opt_{1,\ldots,i}) && (2)\\
&= \Phi(opt_{1,\ldots,i}) - \Phi(opt_{1,\ldots,i-1}) && (3)\\
\sum_i cost_i(s) &\leq \sum_i \Phi(opt_{1,\ldots,i}) - \Phi(opt_{1,\ldots,i-1}) && (4)\\
&= \Phi(opt) - \Phi(\{\}) && (5)\\
&= \Phi(opt) && (6)\\
&= \sum_e \sum_{i=1}^{x_e^*} \frac{c_e(i)}{i} \quad [x_e^* \textit{ is congestion in opt}] && (7)\\
&\leq \sum_e c_e(x_e^*) \cdot (1 + \ldots + \frac{1}{k}) && (8)\\
&= opt \cdot H_k \quad \blacksquare && (9)
\end{aligned}
$$

- Step 1 $\rightarrow$ 2 follows because $\frac{c_e(x)}{x}$ is monotone decreasing in x, so sending some people home will increase the remaining people's costs.
- Step 2 $\rightarrow$ 3 follows because cost-sharing is a potential game. The cost of player i is equal to the difference in Φ when i is sent home. I.e, $\forall$ strategies s and players i, $cost_i(s) = \Phi(s) - \Phi(s_{-i})$.

- In step 4 we add up the inequality over all players.
- Step $4 \to 5$ follows because the expression we came up with in (4) is a telescoping sum.
- Step $5 \to 6$ follows because $\Phi(\{\}) = 0$. Recall that $\Phi = \sum_e \sum_{i=1}^{x_e} \frac{c_e(i)}{i}$
- Step $7 \to 8$ follows because $c_e(x)$ is monotone increasing, so $c_e(x_e^*) \geq c_e(i)$ for all other i.

But note that strong Nash are not guaranteed to exist!