CS 6840	March 9, 2012

Lecture 22: Auction Approximations

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Recap of Last Lecture

Recall that maximizing profit is equivalent to maximizing the virtual surplus in an auction. In other words, the expected revenue is equal to the expected virtual surplus

$$\mathbb{E}[\text{Revenue}] = \mathbb{E}_q\left[\sum_i \phi_i(q)x_i(q)\right],$$

where $\phi_i(q)$ is the marginal revenue

$$\phi_i(q) = R'(q).$$

Call a value distribution F_i *regular* if its corresponding ϕ_i is monotone decreasing in q (or, equivalently, monotone increasing in v). If this is the case, then maximizing surplus is a monotone allocation, so we may use VCG to obtain the optimal revenue mechanism as follows.

1. Solicit bids b_i from players,
2. Translate to the quantile space q_i for each player via the distribution for each player i,
3. Apply surplus maximization with input $\phi_i(q_i)$ for all players i with VCG to get an allocation x and price p',
4. Translate prices back to actual value space, and output the allocation x and prices $\phi_i^{-1}(p'_i)$.

Equivalently, we could compute the allocation and then charge critical prices, which are the prices that make the auction truthful.

Suppose we have a single item auction and that all players have the same i.i.d. distribution. Then the virtual value $\phi_i(q)$ for each player is the same $\phi(q)$. Furthermore, if the distribution is regular, then maximizing virtual surplus is equivalent to maximizing value, so we may give the item to the player with the highest value, conditioned on the virtual surplus being positive.

This process translates to a second prices auction (as second price maximizes value) with all players with $\phi(v) < 0$ thrown out. Throwing out players is equivalent to having a *reserve*, which sets a threshold on the values of the players. Define η, the *monopoly reserve*, to be $\phi^{-1}(0)$. Then since we assumed that the distributions were regular, for all $v > \eta$, $\phi(v) \geq 0$, and for all $v < \eta$, $\phi(v) \leq 0$. Therefore, we may now use a second price auction with reserve η.

Approximations for non-i.i.d. Auctions

Now suppose that we have a single item auction where the distributions are not i.i.d. Now the optimal auction may not be so simple: The virtual value functions are not the same for each player, so maximizing virtual surplus is not the same as maximizing actual surplus. Thus, the optimal auction can be quite complicated, so we will try to form a simple mechanism that is a good approximation of the optimal one. We will focus on a single item auction with non-i.i.d. distributions that are still regular.

An Example

Let's first look at an example of a regular, non-i.i.d. auction. The optimal auction in this example will not be in the form of a second price auction with monopoly reserve. Let there be two bidders for one item. Bidder 1 has a distribution F_1 that is $\mathcal{U}[0, 1]$ (uniformly drawn from the interval $[0, 1]$), and bidder 2 has a distribution F_2 that is $\mathcal{U}[0, 2]$.

Let's prove that these two distributions are regular. First, we claim that the virtual value is

$$\phi_i(v) = v - \frac{1 - F_i(v)}{f_i(v)},$$

where F_i and f_i are the cumulative density and its derivative, the probability density distribution, respectively. By definition, $\phi_i(q) = R_i'(q) = (v(q) \cdot q)'$, and $q_i(v) = 1 - F_i(v)$. The inverse function of q_i is $v_i(q) = F_i^{-1}(1 - q)$. Now by the product rule, we have

$$\phi_i(q) = v_i(q) + q v_i'(q).$$

By the inverse function theorem,

$$(F^{-1}(q))' = \frac{1}{F'(F^{-1}(q))}.$$

Therefore,

$$\phi_i(q) = v_i(q) - \frac{q}{F_i(f_i^{-1}(1-q))},$$

as f_i is the derivative of F_i. Now replace q with $1 - F(v)$, and have

$$\phi_i(v) = v - \frac{1 - F_i(v)}{f_i(v)},$$

as desired.

Now in the context of our auction, we may now directly compute ϕ_1 and ϕ_2.

$$\phi_1(v) = 2v - 1$$
$$\phi_2(v) = 2v - 2$$

The optimal mechanism solicits bids and gives to the player with the highest virtual value. Therefore, bidder 1 wins when $\phi_1(v_1) \geq \max\{\phi_2(v_2), 0\}$, which is equivalent to $\phi_1(v_1) \geq \max\{v_2 - 1/2, 1/2\}$. Thus, we give the item to the first player more often, as we give it to him even if his value is smaller. Therefore, this optimal mechanism does not fit into our second price auction with reserve framework.

An Approximation Scheme

Now let's disregard the optimal mechanism and go back to second price with reserve auctions as an approximation. Define a second price with reserve $\tilde{r} = (r_1, \ldots, r_n)$ to be

1. Solicit bids b_i from players,
2. Throw out players where $b_i < r_i$,
3. Give the item to the player i that maximizes value v_i, and
4. Charge the critical price to player i.

Recall that fixing an allocation fully determines the prices that make the auction truthful, so the last step is superfluous. Note that we allow different reserves for each player. Denote this scheme as SPA-$\tilde{r}$. We now present the main theorem.

Theorem 1. *SPA-$\tilde{\eta}$ (SPA with monopoly reserves) is a 1/2-approximation to the optimal auction.*

Proof. Let OPT be the revenue of the optimal mechanism and APX be the revenue of SPA-$\tilde{\eta}$. We need to show that $APX \geq 1/2 \cdot OPT$. Denote I to be the winner of OPT and J to be the winner of SPA-$\tilde{\eta}$. Then,

$$OPT = \mathbb{E}[\phi_I(v_I)].$$

Now split the expectation into two cases, where either $I = J$ or $I \neq J$.

$$OPT = \mathbb{E}[\phi_I(v_I) \mid I = J]\,\mathbb{P}[I = J] + \mathbb{E}[\phi_I(v_I) \mid I \neq J]\,\mathbb{P}[I \neq J]$$

The goal now is to bound each term by APX. In the first case, where $I = J$,

$$\mathbb{E}[\phi_I(v_I) \mid I = J]\,\mathbb{P}[I = J] = \mathbb{E}[\phi_J(v_J) \mid I = J]\,\mathbb{P}[I = J]$$

Now since the distributions are regular, the monopoly reserve guarantees us that $\phi_J(v_J)$ will always be positive. Therefore, we may add the $I \neq J$ term in freely.

$$\begin{aligned}
\mathbb{E}[\phi_I(v_I) \mid I = J]\,\mathbb{P}[I = J] \leq\ & \mathbb{E}[\phi_J(v_J) \mid I = J]\,\mathbb{P}[I = J] \\
& + \mathbb{E}[\phi_J(v_J) \mid I = J]\,\mathbb{P}[I \neq J] \\
=\ & \mathbb{E}[\phi_J(v_J)] \\
=\ & APX
\end{aligned}$$

The second term is a little more interesting. What precisely does it mean that $I \neq J$? It implies that the winner of the second price auction was not the same as the first price auction. Because player J won in the second price auction, we must have $v_J > v_i$. But since he was not allocated in the optimal auction, $\phi_I(v_I) \geq \phi_J(v_J)$, and player J had smaller virtual value. Note from our previous argument that

$$\phi(v) = v - \frac{1 - F(v)}{f(v)} \leq v.$$

Putting everything together, we have

$$\mathbb{E}[\phi_I(v_I) \mid I \neq J]\,\mathbb{P}[I \neq J] \leq \mathbb{E}[v_I \mid I \neq J]\,\mathbb{P}[I \neq J].$$

Since I was in the second price auction but J won and second price auctions are truthful, we must have $p_J \geq v_I$.

$$\mathbb{E}[v_I \mid I \neq J]\,\mathbb{P}[I \neq J] \leq \mathbb{E}[p_J \mid I \neq J]\,\mathbb{P}[I \neq J]$$

Now since prices are positive,

$$\begin{aligned}
\mathbb{E}[p_J \mid I \neq J]\,\mathbb{P}[I \neq J] &\leq \mathbb{E}[p_J \mid I \neq J]\,\mathbb{P}[I \neq J] + \mathbb{E}[p_J \mid I = J]\,\mathbb{P}[I = J] \\
&= \mathbb{E}[p_J] \\
&= APX.
\end{aligned}$$

$\square$