CS 6840 – Algorithmic Game Theory (3 pages) Spring 2012

Lecture 40: Brouwer fixed point theorem

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Brouwer fixed point theorem is used in proving the existence of Nash equilibrium. We start with 2-dimensional space. Consider a simplex in this space represented by

$$\Delta = \{(x, y, z) \geq 0, x + y + z = 1\}$$

and also a continuous function $f : \Delta \to \Delta$. We claim that there exists an x such that $f(x) = x$.

The main tool in proving this is Sperner lemma, which is a simple combinatorial lemma. We will prove this lemma first, then try to connect it with the Brouwer theorem. It talks about the triangulation of the simplex. The simplex is divided as shown in Figure 1, and a coloring is a choice of colors for each of its points.

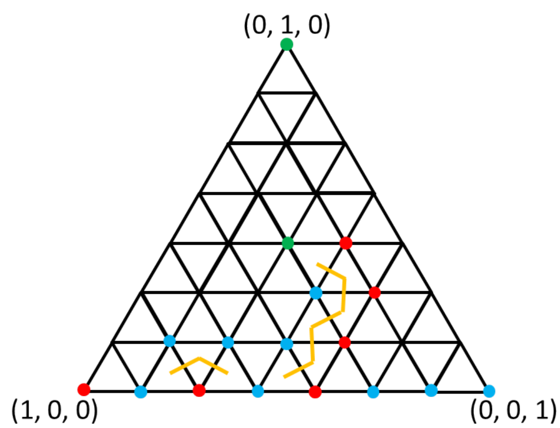

Figure 1: Coloring on 2-dimensional simplex.

Theorem 1 (Sperner lemma)**.** If the coloring satisfies that the corners have different colors, and each point on a side uses one of the colors from corner on the side, there exists a multi-colored (all corners have different colors) little triangle.

More generally, for high-dimensional simplex, there exists a little simplex with all corners having different colors.

We want to prove this by induction, it helps if we make the theorem stronger. Thus we want to prove that there exist odd number of little triangles (or little simplexes for high dimensional space). Today we only prove for the 2-dimensional space.

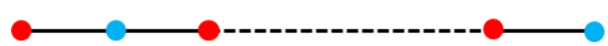

Figure 2: This is a figure.

Proof. Let us first think of 1-dimensional case (See Figure 2). The simplex is just a line. We assume one corner is red and the other is blue. The existence is easy to see, since one side is red and the other side is blue, it has to switch at some point. Now we want to prove the number of red-blue segments is odd. This is true because if we start with the red corner and have even number of red-blue segments, the other color will have the same color red. So the 1-dimensional case (base case) is done.

For 2-dimensional simplex, we start off the red and blue side. By induction, we know that the number of red-blue segments on this side is odd. Now we define a walk on little triangles starting on the red-blue side: we enter on a red-blue side, if the little triangle is multi-colored, this walk reaches the end. Else, the missing color must be red or blue. In either case, we have another red-blue side, and we just leave on this side and continue walking (See the yellow lines in Figure 1). The walk starts on red-blue side can end either in multi-colored triangle, or leave the big triangle on a little red-blue segments. Note these lines of walk are disjoint as participating triangles have 1 or 2 red-blue sides. They cannot have three, so the lines won't merge. Since there are odd number of red-blue sides, they cannot all pair up. There must exist one red-blue side so the walk off it ends in a multi-colored triangle. So far we proved the existence.

To prove that there exist odd number of multi-colored triangles, we consider not only the walk off the red-blue segment on the big side, but also the walk off the red-blue side of a multi-colored triangle. This walk will end either on the big red-blue side, or on a multi-colored triangle. Since the lines of the walk won't merge, they just pair up either multi-colored triangle, or red-blue segments on the side. By induction, there are odd number of red-blue segments on the side, so there must be odd number of multi-colored triangles. □

It will be the same kind of argument if we want to lift it up to high-dimensional space. Now we want to use the lemma in 2-dimensional space to prove the existence of fixed point.

Let $f : \Delta \to \Delta$ be a continuous function, by continuous we usually mean $\forall p \in \Delta$, $\forall \epsilon$, $\exists \delta$ so that if $||p-q|| \leq \delta$, then $||f(p)-f(q)|| \leq \epsilon$, where the norm can be chosen freely. However, here we would like to use a stronger statement: $\forall \epsilon$, $\exists \delta$ so that $\forall p, q$, if $||p-q|| \leq \delta$ then $||f(p)-f(q)|| \leq \epsilon$. Using some calculus, it is not hard to prove that if Δ is bounded and closed, the original continuity definition implies the stronger one. Here we just use it without proving. We want to prove that there is a fixed point under the stronger version of continuity.

Now we need a meaningful coloring scheme. Assume

$$
\begin{aligned}
p &= (x, y, z) \\
f(p) &= (x', y', z')
\end{aligned}
$$

We color the point

Red if $x' < x$

Green if $y' < y$ and $x' \geq x$

Blue if $z' < z$ and $x' \geq x$ and $y' \geq y$

Intuitively, we use the coordinate that gets smaller to give it a color. We can show that this coloring is proper that it satisfies Sperner coloring rule. We first want to examine the three corners $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. They are colored with red, green and blue respectively. It is easy to see that this coloring satisfies the rule since if they are not fix points, the only coordinate that can get smaller is the one with 1 (0 cannot get any smaller). Second we want to claim the lines between the corners also obey the coloring rule. The lines are just the side, and the point on which must have one zero coordinates. The coordinates just corresponds the color of the opposite corner. Since it cannot get smaller, it cannot take that color.

Now we want to have some intuition of the following proof. Roughly speaking, a (small) multi-colored triangle will be mapped to to an (small) area due to continuity. The three different colors mean that each one of the three point will get one coordinate smaller respectively. However, they cannot all get smaller because of the sum to one. So none of them shall get at least much bigger. We will continue next time.