11 Sep 2023

Algebraic Algorithms for Bipartite Matching

Announcement. Billy Jin, "Advice-Augmented Algorithms for Online Matching and Resource Allocation"

Gates 114, 3:45 pm, today

Permanent and Determinant

If $A = (A_{ij})$ is an $n \times n$ square matrix

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

S_n ← the set of permutations of $[n]$.

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$\mathrm{sgn}(\sigma)$: $=0$ if σ is product of an even # transpositions; $=1$ otherwise

If $G = (L, R, E)$ is bipartite with

$$L = \{u_1, \dots, u_n\} \qquad R = \{v_1, \dots, v_n\}$$

Let

$$A_{ij} = \begin{cases} 1 & \text{if } (u_i, v_j) \in E \\ 0 & \text{if } (u_i, v_j) \notin E \end{cases}$$

Then

$$\prod_{i=1}^{n} A_{i,\sigma(i)} = \begin{cases} 1 & \text{if } \{(u_i, v_{\sigma(i)})\} \text{ is a perfect matching} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathrm{per}(A) = \#\text{ of perfect matchings in } G.$$

#P: the complexity class whose complete problems are the two sides of this equation.

$\det(A)$ = a weird quantity that "counts" perfect matchings "with cancellations."

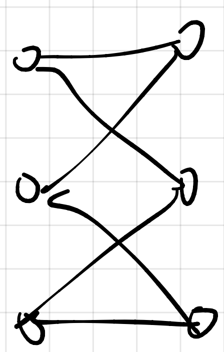
Nevertheless $\det(A) \bmod 2$ tells us if G has an even or odd # of perfect matchings.

Consider another matrix B associated with G,

a formal variable

$$B_{ij} = \begin{cases} x_{ij} & \text{if } (u_i, v_j) \in E \\ 0 & \text{if } (u_i, v_j) \notin E \end{cases}$$

G



$$B = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}$$

Observe $\det(B) \neq 0$ if and only if G has a perfect matching.

Lovász's Algorithm for Perfect Matching Decision Problem

Form the matrix B.

Let $\mathbb{F}$ be a field with at least n^2/δ elements.

Substitute independent uniformly random elements of $\mathbb{F}$ for the variables in B. Call this $\hat{B}$.

Evaluate $\det(\hat{B})$.

If $\det(\hat{B}) = 0$, output "no perfect matching."

Else output "a perfect matching exists."

This also has no false positives. If it says a perfect matching exists, there really is one.

It can have false negatives. Try to prove Pr(false negative) is low.

Schwartz-Zippel Lemma If $P(X_1, \dots, X_m)$ is a non-zero multivariate degree d polynomial over a field, $\mathbb{F}$, and S is a subset of $\mathbb{F}$ with s elements, and we subsitute random, indep. elements of S for $X_1, \dots, X_m$

$$\Pr(P \text{ evaluates to zero}) \leq \frac{m \cdot d}{s}$$

Proof. (induction on m) if $m=1$, P is a nonzero degree d polynomial, it has $\leq d$ roots. So $\Pr(P \text{ eval's to zero}) \leq d/s$.

For $m > 1$,

$$P(X_1, \dots, X_m) = \sum_{i=0}^{d} Q_i(X_1, \dots, X_{m-1}) \cdot X_m^i$$

For some $i \in \{0, \dots, d\}$ $Q_i \neq 0$.

Let c be greatest such i.

When we substitute random $X_1, \dots, X_m$

Case 1. $Q_c(X_1, \dots, X_{m-1}) = 0$.

$\Pr(\text{this event}) \leq \frac{(m-1) \cdot d}{s}$ (IND HYP)

Case 2. $Q_c(X_1, \dots, X_{m-1}) \neq 0$ but

$$P(X_1, \ldots, X_m) = 0.$$

That can only happen if X_m is one of the roots of

$$\sum_{i=0}^{c} Q_i(X_1, \ldots X_{m-1}) X_m^i$$

There are at most c roots.

$\Pr(\text{this event}) \leq \frac{c}{S} \leq \frac{d}{S}.$

<u>Case 3.</u> $P(X_1, \ldots, X_m) \neq 0.$

Sum probabilities of Case 1, Case 2 $\Rightarrow$ Q.E.D.