6 Sep 2023 — Online Matching

Announcement: problem (2b) is modified; see Ed Discussions.

If still waitlisted: stay tuned, I am emailing Student Services today.

$G = (L, R, E)$

$L = \{$vertices always present in $G\}$

e.g. time slots in a calendar

$R = \{$vertices that arrive one at a time$\}$

Upon arrival, each vertex in R specifies its set of neighbors and must be matched to one of them (or left unmatched) irrevocably before the next arrival.

Objective: maximize the number of edges in the matching.

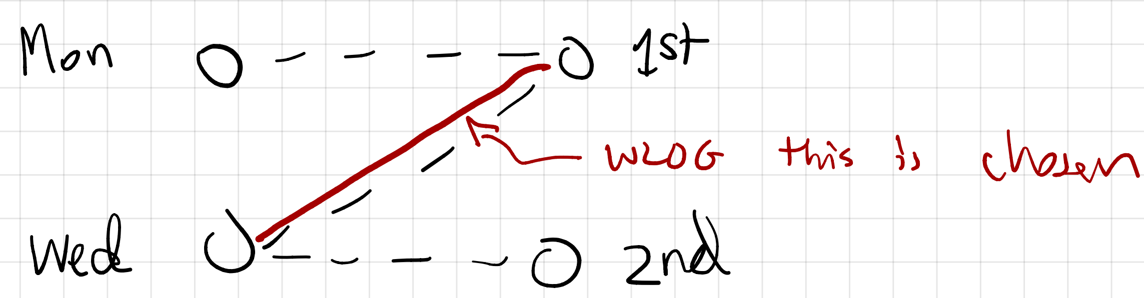
An algorithm is c-competitive if it is always within a factor c of the optimum on every input instance.

i.e. letting ALG = # edges in the algo's matching

OPT = # edges in max matching

$$OPT \le c \cdot ALG$$

Deterministic alg's can't be better than 2-competitive.



OPT = 2
ALG = 1

A randomized alg. on this graph can toss a coin, and get
$$ALG = \begin{cases} 1 & \text{w. prob. } 1/2 \\ 2 & \text{w. prob. } 1/2 \end{cases}$$

$$\left.\begin{array}{l} \mathbb{E}[ALG] = 3/2 \\ OPT = 2 \end{array}\right\} \tfrac{4}{3}\text{-competitive,}$$
on this particular graph.

Def. An algorithm for online matching is <u>greedy</u> if it always finds a match for each vertex that has at least one free neighbor when it arrives.

<u>Prop.</u> Any greedy online matching algorithm is 2-competitive.

<u>Proof.</u> Say the algo. outputs M and M^* is the max matching.

Map M^* to M as follows.

For $(u,v) \in M^*$ and $(u,v) \in M$ then
$$f(u,v) = (u,v).$$

For $(u,v) \in M^*$ and $(u,v) \notin M$ but $(u',v) \in M$
$$f(u,v) = (u',v).$$

For $(u,v) \in M^*$, $(u,v) \notin M$, v is free in M.
Then $\exists$ $(u,v') \in M$ by greedy property.

$$f(u,v) = (u,v')$$

We've constructed $f: M^* \to M$ with the property $\forall e \in M^*$, e and $f(e)$ have at least one endpoint in common.

For every $e' \in M$, $f^{-1}(e')$ has at most two elements: an edge touching left endpoint of e' and one touching the right endpoint.

$\therefore$ $|M^*| \leq 2 \cdot |M|$ as claimed.

The RANKING algorithm of Karp, Vazirani, Vazirani.

1. Sample a uniformly random total ordering of L
2. Whenever $v \in R$ arrives, if it has at least one free neighbor, match to the one that comes earliest in this ordering.

LP relaxation of max bipartite matching:

$$\max \sum_{(u,v) \in E} x_{uv}$$

$$\text{s.t.} \quad (y_u) \quad \sum_{v} x_{uv} \leq 1 \qquad \forall u \in L$$

$$(y_v) \quad \sum_{u} x_{uv} \leq 1 \qquad \forall v \in R$$

$$x_{uv} \geq 0 \qquad \forall (u,v)$$

Scale inequalities by scale factors

y_u, y_v, you derive

$$\sum_{u \in L} y_u \left[\sum_{v} x_{uv} \right] + \sum_{v \in R} y_v \left[\sum_{u} x_{uv} \right] \leq \sum_{u} y_u + \sum_{v} y_v$$

$$\sum_{(u,v) \in E} (y_u + y_v) x_{uv}$$

as long as $y_u, y_v \geq 0$.

This upper bounds OPT as long as

$y_u + y_v \geq 1 \quad \forall (u,v) \in E$.