$\boxed{\text{1 Sep 2023}}$ LP relaxation of bipartite min-cost matching

WTF?

Min-cost matching, restated algebraically.

Variable x_{uv} intended meaning $x_{uv} = \begin{cases} 1 & \text{if } (u,v) \in M \\ \emptyset & \text{otherwise} \end{cases}$

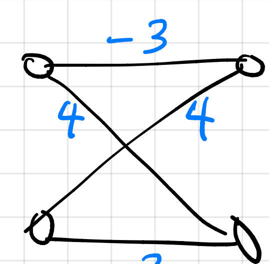
$$\min \sum_{u \in L, v \in R} c(u,v) \cdot x_{uv}$$

$$\text{s.t.} \quad \sum_{v} x_{uv} = 1 \quad \forall u \in L$$

$$\sum_{u} x_{uv} = 1 \quad \forall v \in R$$

$$x_{uv} \in \{0,1\}$$

IF $x_{uv} \notin \{0,1\}$, then e.g.



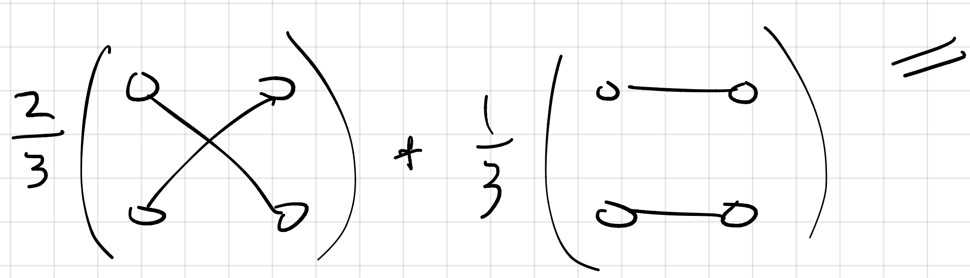
$$\min \sum_{u \in L, v \in R} c(u,v) \cdot x_{uv}$$

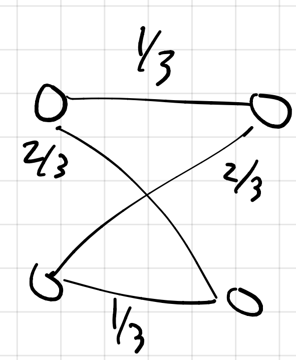
$$\text{s.t.} \quad \sum_{v} x_{uv} = 1 \quad \forall u \in L$$

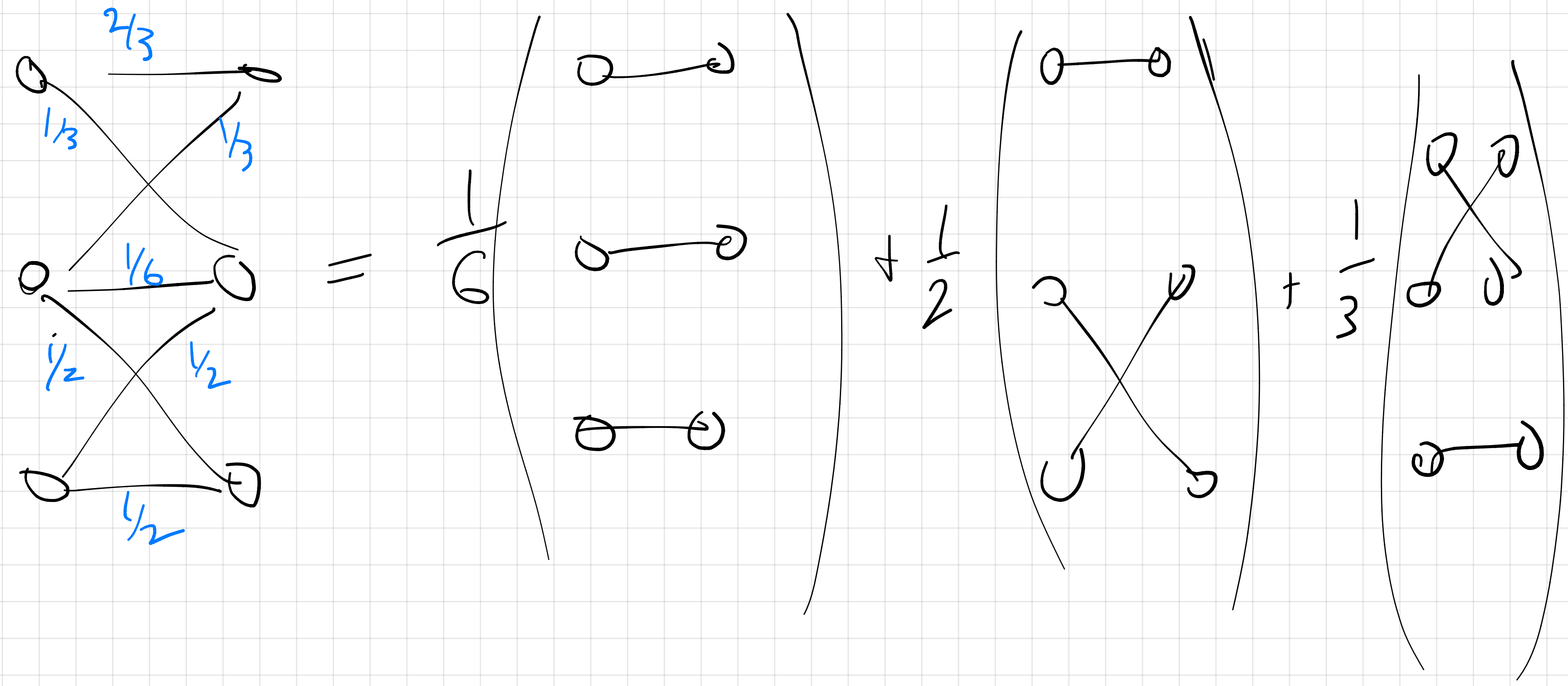
$$\sum_{u} x_{uv} = 1 \quad \forall v \in R$$

$$x_{uv} \geq 0$$

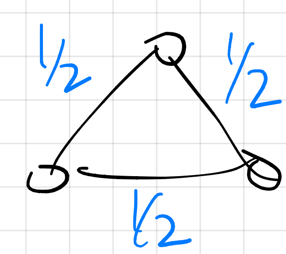
A solution of these constraints is called a fractional perfect matching.





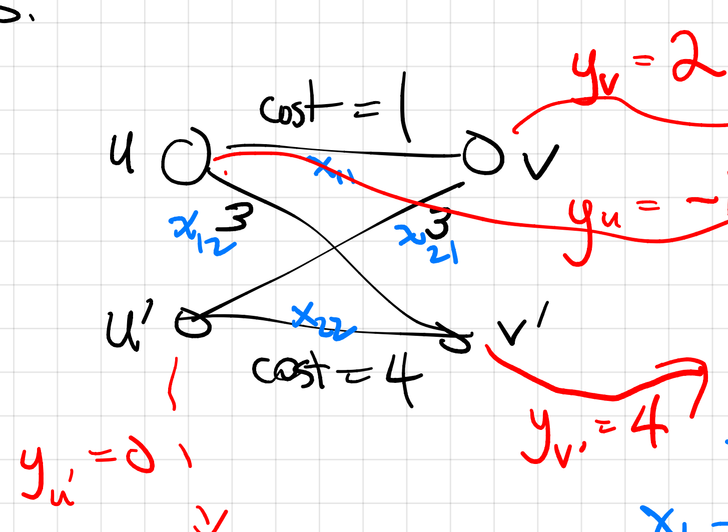


<u>Birkhoff–von Neumann Thm.</u> Every "fractional perfect matching" in a bipartite graph is a convex combination of perfect matchings.



How would one show that a specified perf. matching has minimum cost among <u>all</u> fractional perfect matchings?

E.g.



$$2x_{11} + 2x_{21} = 2$$
$$-x_{11} - x_{12} = -1$$
$$4x_{12} + 4x_{22} = 4$$

$$x_{11} + 2x_{21} + 3x_{12} + 4x_{22} = 5$$

$y_u + y_v = c(u,v)$ when $(u,v) \in M$ $\qquad$ $y_{u'} + y_v \le c(u',v)$

$$x_{11} + 3x_{21} + 3x_{12} + 4x_{22} \geq 5$$

<u>Exercise:</u> For the min-cost k edge matching problem complete this story by expressing it algebraically and explaining where properties (3)–(4) come from.